In The Name of Allah

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Scientific Committee:

Organizing Committee:

Topics of the Workshop:

- Statistical methods in reliability
- Nonparametric methods in reliability
- Accelerated life testing
- Repairable and maintenance models
- Reliability of coherent systems
- Ordered data in reliability
- Stochastic orders in reliability
- Stress-Strength Models
- Reliability and information theory
- Ageing concepts
- Dependency in reliability

Preface

On behalf of the organizing and scientific committees, we would like to extend a very warm welcome to all the participants of the 2th Workshop on Reliability and its Applications.

We hope this seminar provide an environment for useful discussions and also to exchange scientific ideas and opinions.

The program of this seminar is organized in 6 key speakers, 20 oral contributions, and 8 poster presentations.

We wish to express my gratitude to the numerous individuals (especially the chancellor of the Isfahan University) and organizations that have contributed to the success of this workshop, in which more than 85 colleagues, researchers, and postgraduate students have participated.

Finally, we would like to extend our sincere gratitude to the students of the Faculty of Mathematical Sciences at Isfahan University for their kind help and cooperation. We wish you all every success.

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The Dynamic Signature of Coherent Systems

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Abstract

The concept of 'signature' is a useful tool to study the stochastic and aging properties of coherent systems. We consider a coherent system, and assume that there is some partial information about the failure status of the system lifetime. We study various properties of the conditional signature.
Keywords and Phrases: Order statistics, stochastic order, residual lifetime, inactivity time.
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1 Introduction

A system consisting of n components is said to be a coherent system if it has no irrelevant components (a component is said to be irrelevant if its performance does not effect the performance of the system), and the structure function of the system is monotone in every component (that is, when a failed component is replaced by a working component the working system does not fail). In recent years, researchers have studied various aspects of the reliability and aging specifications of coherent systems. Asadi and Bayramoglu [1] studied the mean residual life function of k-out-of-n structures at the system level. Tavangar and Asadi [15] investigated the mean inactivity time of the components of $(n-k+1)$-out-of-n systems at the system level. Khaledi and Shaked [6] obtained signature-based stochastic comparisons between the residual lifetimes of coherent systems. Asadi and Goliforushani [2] explored aging properties of coherent systems when the vector of the signature satisfies some conditions. Li and Zhang [8] obtained stochastic ordering results on conditional coherent systems on the basis of the signature concept. Eryılmaz [3] explored the properties of the residual lifetime of linear consecutive k-out-of-n systems based on the definition of signature. Eryılmaz and Zuo [4] compute the signature of a system with two common failure criteria.

Suppose that a coherent system with n components has statistically independent and identically distributed lifetimes $X_1, X_2, ..., X_n$ with a common continuous distribution F. Let $T = T(X_1, ..., X_n)$ denote the lifetime of the system. The signature of the system is defined to be a probability vector $\mathbf{s} = (s_1, s_2, ..., s_n)$ such that

$$s_i = P(T = X_{i:n}), \quad i = 1, 2, ..., n. \tag{1}$$

It can be shown that $s_i = n_i/n!$, where n_i is the number of ways that distinct $X_1, X_2, \ldots, X_n$ can be ordered such that $T(x_1, ..., x_n) = x_{i:n}$, $i = 1, ..., n$, where $x_{i:n}$ is the value of $X_{i:n}$; see Samaniego [3].

The probability s_i is in fact the probability that the component with lifetime $X_{i:n}$ causes the failure of the system. As the signature vector $\mathbf{s}$ does not depend on the common distribution function of X_i, the survival function of T, denoted by $\bar{F}_T(t)$, can be written as a mixture of the survival functions of $X_{i:n}$, denoted by $\bar{F}_{i:n}(t)$ $i = 1, 2, ..., n$, with weights $s_1, s_2, ..., s_n$. In other words, $\bar{F}_T(t)$ can be represented as

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t). \tag{2}$$

This result enables one to study the stochastic and aging properties of a coherent system based on the properties of its signatures. Kochar et al. [2] showed that, for two coherent systems with lifetimes T_1 and T_2, if the signature vector of the first system is stochastically less than the signature vector of the second one, then T_1 is also stochastically less than T_2. Navarro et al. [9]-[11] used the concept of signature to compare the reliability of coherent systems when the components are not necessarily statistically independent. In a recent study, Navarro et al. [12] introduced the concept of joint signature for coherent systems with shared components (see also, Navarro [13] for more developments on the reliability properties of coherent systems).

In reliability engineering, usually the operators of the systems may have some partial information about the lifetime of the system, and based on that information they may be interested in getting information about the failure probability of the system. For example, an operator may know that, at time $t > 0$, the system is still operating, i.e. $T > t$, or the exact failure time of the system is t, i.e. $T = t$. In this context, it could be interesting for the operator to investigate which component of the system is more probable to cause the failure of the system.

This paper is an investigation of the properties of the signature of a coherent system under the condition that, at time t, some information about the status of the system lifetime is available. In a general setup, we assume that the lifetime of the system T is in a subset of the positive real line, i.e, $T \in A \subseteq [0, \infty)$. Then, we are interested in investigating the properties of the conditional probability

$$p_i(A) = P(T = X_{i:n} | T \in A) \quad i = 1, 2, ..., n.$$

$p_i(A)$ is the probability that the component with lifetime $X_{i:n}$ causes the failure of the system under the condition that the system lifetime is known to be in the set A. Obviously, when $A = [0, \infty)$, we have $p_i(A) = s_i$, $i = 1, 2, ..., n$, where s_i is the ith element of the signature of the system given in (1). There are three choices for set A which are of more particular interest.

(i) the set $A = \{t\}$ corresponds to the immediate failure of the system. For this choice of A, the operator knows that the exact failure time of the system is t.

(ii) the set $A = (t, \infty)$ corresponds to the residual lifetime of the system. That is, for this choice of A, the operator knows that the system is still working at time t.

(iii) the set $A = [0, t)$ corresponds to the inactivity time of the system. For this choice of A, the operator knows that the system has failed sometime before time t.

Recently, Navarro et al. [10] (see also Zhang [16], and Zhang and Yang [18]) have obtained some results on $p_i(A)$ when $A = (t, \infty)$. Goliforushani and Asadi [5] (see also Zhang [17]) have investigated some properties of $p_i(A)$ when $A = (0, t)$.

In Section 2, we first derive the form of the conditional probability $p_i(A)$ for an arbitrary subset A of the positive real line. Then, we explore some properties of $p_i(A)$ where A is assumed to be in the forms of parts (i)-(iii). In the sequel, we will study the behavior of $p_i(A)$ in each part when t tends either to infinity, or tends to zero. And we will show that, for two coherent systems, if the components of the systems are stochastically ordered, then the corresponding conditional signatures of the systems are also stochastically ordered.

2 Main Results

Consider a coherent system consisting of n statistically independent and identically distributed components with ordered lifetimes $X_{1:n}, X_{2:n}, ..., X_{n:n}$, and signature vector $\mathbf{s} = (s_1, s_2, ..., s_n)$. Let $A \subseteq [0, \infty)$. Then, provided that $P(T \in A) > 0$, we have

$$\begin{aligned} p_i(A) &= P(T = X_{i:n}|T \in A) \\ &= \frac{P(T = X_{i:n}, T \in A)}{P(T \in A)} \\ &= \frac{P(T \in A|T = X_{i:n})P(T = X_{i:n})}{P(T \in A)} \\ &= \frac{P(X_{i:n} \in A|T = X_{i:n})P(T = X_{i:n})}{P(T \in A)} \\ &= \frac{s_i P(X_{i:n} \in A)}{\sum_{j=1}^{n} s_j P(X_{i:n} \in A)}, \quad i = 1, 2, ..., n, \end{aligned} \tag{3}$$

where the last equality follows from the fact that events $\{T = X_{i:n}\}$ (which does not depend on the underlying distribution function) and $\{X_{i:n} \in A\}$ are statistically independent.

The probability $p_i(A)$ can be interpreted in a Bayesian point of view. Assume that the probability vector $\mathbf{s} = (s_1, s_2, ..., s_n)$ denotes a prior distribution in which s_i, $i = 1, 2, ..., n$ shows the probability that the system lifetime is equal to the ordered lifetime $X_{i:n}$ before the system is put in operation. This result can be viewed as the prior belief of the system designer about the probability of the failure time of the system when the system has not yet been used. Now assume that the system is put in operation at time $t = 0$, and the designer has the information (data) that, for example, at time $t > 0$, the system is still working; that is to say $T > t$. Under these circumstances, the probability $p_i(A)$, where $A = (t, \infty)$, can be considered as the posterior belief (posterior distribution) of the designer on the probability of failure time of the system given that the system is alive at time t. In the following, we consider the three different cases (i), (ii), and (iii).

(i) The first case which we are interested in is the set $A = \{t\}$. In this case, $p_i(A)$ is given as

$$p_i(t) = P(T = X_{i:n}|T = t), \quad i = 1, 2, \ldots, n.$$

Under the assumption that F is absolutely continuous with density function f, and survival function $\bar{F} = 1 - F$, we can show that $p_i(t)$, $i = 1, 2, ..., n$, is given as

$$p_i(t) = \frac{s_i f_{i:n}(t)}{\sum_{j=1}^{n} s_j f_{j:n}(t)}, \quad t > 0, \tag{4}$$

where

$$f_{i:n}(t) = \frac{n!}{(i-1)!(n-i)!} f(t) F^{i-1}(t) \bar{F}^{n-i}(t), \quad t > 0$$

is the density function of $X_{i:n}$. Denote $\phi(t) = F(t)/\bar{F}(t)$ for t such that $\bar{F}(t) > 0$, the odds of the event that a component has a lifetime less than t. Then,

$$p_i(t) = \frac{c_{i,n} s_i \phi^{i-1}(t)}{\sum_{j=1}^{n} c_{j,n} s_j \phi^{j-1}(t)}, \quad i = 1, 2 \ldots, n, \tag{5}$$

where $c_{i,n} = \frac{n!}{(i-1)!(n-i)!}$, $i = 1, 2, \ldots, n$. We call the dynamic vector

$$\mathbf{p}(t) = (p_1(t), p_2(t), ..., p_n(t))$$

as the vector of the signature associated with the immediate failure of the system. Note that $p_i(t)$ does not depend on the density function f.

In the sequel, we assume that Y is a discrete random variable with probability mass function $p_i(t)$, and denote its survival function by $\bar{P}_j(t)$, i.e. we assume that $p_i(t) = P(Y = i)$, $i = 1, 2, ..., n$, and $\bar{P}_j(t) = P(Y \geq j) = \sum_{k=j}^{n} p_k(t)$, $j = 1, 2, \ldots, n$.

(ii) The second case that we consider for A is $A = (t, \infty)$, which is related to the residual lifetime of the system. In this case, $p_i(A)$, which we denote it by $q_i(t)$, $i = 1, 2, \ldots, n$, is

$$\begin{aligned} q_i(t) &= P(T = X_{i:n} | T > t) \\ &= \frac{s_i \bar{F}_{i:n}(t)}{\sum_{j=1}^{n} s_j \bar{F}_{j:n}(t)}, \end{aligned} \tag{6}$$

where

$$\bar{F}_{i:n}(t) = \sum_{k=0}^{i-1} \binom{n}{k} F^k(t) \bar{F}^{n-k}(t).$$

The dynamic vector $\mathbf{q}(t) = (q_1(t), q_2(t), ..., q_n(t))$ is called the vector of the residual signature of the system. It is easy to show that $q_i(t)$ can be represented in terms of $\phi(t)$ as

$$q_i(t) = \frac{s_i \sum_{m=0}^{i-1} \binom{n}{m} \phi^m(t)}{\sum_{j=1}^{n} s_j \sum_{m=0}^{j-1} \binom{n}{m} \phi^m(t)}, \quad i = 1, 2, \ldots, n.$$

We assume that Y_1 is a discrete random variable with probability mass function $q_i(t) = P(Y_1 = i)$, $i = 1, 2, ..., n$, and denote the corresponding survival function by $\bar{Q}_j(t)$, i.e. $\bar{Q}_j(t) = P(Y_1 \geq j) = \sum_{k=j}^{n} q_k(t)$, $j = 1, 2, \ldots, n$.

(iii) The third set that we are interested in is $A = (0, t)$, which corresponds to the inactivity time of the system. The conditional probability $p_i(A)$, which we denote by $r_i(t)$, is

$$\begin{aligned} r_i(t) &= P(T = X_{i:n} | T < t) \\ &= \frac{s_i F_{i:n}(t)}{\sum_{j=1}^{n} s_j F_{j:n}(t)}, \quad i = 1, 2, ..., n, \end{aligned} \tag{7}$$

where

$$F_{i:n}(t) = \sum_{k=i}^{n} \binom{n}{k} F^k(t) \bar{F}^{n-k}(t).$$

We call the dynamic vector $\mathbf{r}(t) = (r_1(t), r_2(t), ..., r_n(t))$ the vector of inactivity signature of the system. Note that $r_i(t)$ can be represented in terms of $\phi(t)$ as

$$r_i(t) = \frac{s_i \sum_{m=i}^{n} \binom{n}{m} \phi^m(t)}{\sum_{j=1}^{n} s_j \sum_{m=j}^{n} \binom{n}{m} \phi^m(t)}, \quad i = 1, 2, \ldots, n.$$

In what follows, we assume that Y_2 is a discrete random variable with probability mass function $r_i(t)$, and denote the corresponding survival function by $\bar{R}_j(t)$, i.e. $\bar{R}_j(t) = \sum_{k=j}^{n} r_k(t)$, $j = 1, 2, \ldots, n$.

Note that the signature $\mathbf{s}$ does not depend on the underlying distribution function F. However, it is clear that the dynamic vectors $\mathbf{p}(t), \mathbf{q}(t)$, and $\mathbf{r}(t)$ depend on the distribution function F.

In the next theorem, we assume that the signature of the system takes the form

$$\mathbf{s} = (0, \ldots, 0, s_i, s_{i+1}, \ldots, s_{j-1}, s_j, 0, \ldots, 0), \quad i = 1, 2..., j, \quad j = 1, 2, .., n, \tag{8}$$

where $s_k > 0$, $k = i, ..., j$.

Table I presents some coherent systems consisting of 4 components, their corresponding signature which is of the form (4), and their corresponding dynamic signature vector $\mathbf{p}(t)$. Some examples of the residual, and inactivity signatures of the systems of order three are given in Navarro et al. [10], and Goliforushani and Asadi [5], respectively.

Theorem 2.1. *Let T be a coherent system signature vector of the form (4). Then*

(a) *for $k = i$, $p_k(t)$ is a non-increasing function of t;*

Table I
Coherent systems of order 4, and their corresponding signatures

$T=\phi(X_1,...,X_4)$	$\mathbf{s}=(s_1,s_2,s_3,s_4)$	$\mathbf{p}(t)=((p_1(t),p_2(t),p_3(t),p_4(t))$
$X_{1:4}=\min(X_1,X_2,X_3,X_4)$	(1,0,0,0)	(1,0,0,0)
$X_{2:4}$ (2-out-of-4)	(0,1,0,0)	(0,1,0,0)
$\min(X_1,\max(X_2,X_3,X_4))$	$(\frac{1}{4},\frac{1}{4},\frac{1}{2},0)$	$(\frac{2}{2+3\phi(t)+6\phi^2(t)},\frac{3\phi(t)}{2+3\phi(t)+6\phi^2(t)},\frac{6\phi^2(t)}{2+3\phi(t)+6\phi^2(t)},0)$
$\max(\min(X_1,X_2),\min(X_3,X_4))$	$(0,\frac{2}{3},\frac{1}{3},0)$	$(0,\frac{2}{2+\phi(t)},\frac{\phi(t)}{2+\phi(t)},0)$
$\min(\max(X_1,X_2),\max(X_3,X_4))$	$(0,\frac{1}{3},\frac{2}{3},0)$	$(0,\frac{1}{1+2\phi(t)},\frac{2\phi(t)}{1+2\phi(t)},0)$
$\max(X_1,\min(X_2,X_3,X_4))$	$(\frac{1}{2},\frac{1}{4},\frac{1}{4},0)$	$(\frac{4}{4+3\phi(t)+3\phi^2(t)},\frac{3\phi(t)}{4+3\phi(t)+3\phi^2(t)},\frac{3\phi^2(t)}{4+3\phi(t)+3\phi^2(t)},0)$
$X_{3:4}$ (3-out-of-4)	(0,1,0,0)	(0,1,0,0)
$X_{4:4}=\max(X_1,X_2,X_3,X_4)$	(0,0,0,1)	(0,0,0,1)

(b) *for $k=j$, $p_k(t)$ is a non-decreasing function of t; and*

(c) *for $i\le k\le j$,*

$$\lim_{t\to 0} p_k(t)=\begin{cases}1 & k=i\\ 0 & k\neq i,\end{cases}$$

and

$$\lim_{t\to\infty} p_k(t)=\begin{cases}1 & k=j\\ 0 & k\neq j.\end{cases}$$

Proof: Under the assumptions of the theorem, $p_k(t)$ can be written as

$$\begin{aligned} p_k(t) &= \frac{c_{k,n}s_k\phi^{k-1}(t)}{\sum_{l=i}^{j}c_{l,n}s_l\phi^{l-1}(t)},\\ &= \frac{c_{k,n}s_k}{\sum_{l=i}^{j}c_{l,n}s_l\phi^{l-k}(t)}, \quad k=i,...,j. \end{aligned} \tag{9}$$

Based on the fact that $\phi(t)$ is non-decreasing in t, see that, for $k=i$, $p_i(t)$ is a decreasing function of t; and for $k=j$, $p_j(t)$ is a non-decreasing function of t. This proves parts (a) and (b). To prove part (c), note that, if $t\to 0$, then $\phi(t)\to 0$, and hence from (9) we get for $k=i$, $\lim_{t\to 0}p_k(t)=1$, and zero otherwise. Using the same argument, if $t\to\infty$, then $\phi(t)\to\infty$, and hence we obtain $\lim_{t\to\infty}p_k(t)=1$ for $k=j$, and zero otherwise. □

For a coherent system with signature vector of the form (4), under the condition that the system fails at time t, part (a) of the theorem shows that, when t gets larger, the probability that the ith ordered lifetime causes failure of the system gets smaller. Part (b) of the theorem indicates that, under the condition that the system fails at time t, when t gets larger, then the probability that the jth ordered lifetime causes the failure of the system gets larger. The first limit in part (c) of the theorem shows that, if the system fails at an early time of its operation, then it is more probable that its failure cause is based upon the component with lifetime $X_{i:n}$. The second limit shows, in long time operation, it is more probable that the system failure cause be the component with lifetime $X_{j:n}$.

One can verify that the analogous results are valid for $q_i(t)$, and $r_i(t)$ (see also, Navarro et al. [10]). We have summarized the results in Table II.

Theorem 2.2. *Let the signature of the system be $\mathbf{s}=(s_1,s_2,...,s_n)$. Then the distribution function $F(t)$ can be recovered from $p_i(t)$ as*

$$\bar{F}(t)=\frac{(n-i)s_{i+1}p_i(t)}{is_ip_{i+1}(t)+(n-i)s_{i+1}p_i(t)}, \quad t>0 \quad i=1,...,n.$$

Proof: It is easily seen that, when the ratio is defined, we have

$$\frac{p_{i+1}(t)}{p_i(t)}=\frac{(n-i)}{i}\frac{s_{i+1}}{s_i}\phi(t), \quad i=1,2,...,n-1.$$

Table II

The time behavior of dynamic signatures, based on signature (4)

$p_i(t)$ is non – increasing in t $\lim_{t\to 0} p_k(t)=1 \quad k=i$ $=0 \quad k\neq i$	$p_j(t)$ is non – decreasing in t $\lim_{t\to\infty} p_k(t)=1 \quad k=j$ $=0 \quad k\neq j$
$q_i(t)$ is non – increasing in t $\lim_{t\to 0} q_k(t)=s_i \quad k=i$ $=0 \quad k\neq i$	$q_j(t)$ is non – decreasing in t $\lim_{t\to\infty} q_k(t)=1 \quad k=j$ $=0 \quad k\neq j$
$r_i(t)$ is non – increasing in t $\lim_{t\to 0} r_k(t)=1 \quad k=i$ $=0 \quad k\neq i$	$r_j(t)$ is non – decreasing in t $\lim_{t\to\infty} r_k(t)=s_j \quad k=j$ $=0 \quad k\neq j$

Because $\phi(t)=F(t)/\bar{F}(t)$, after some simplification, we get the result. □

From the proof of the theorem, it can be also concluded that the ratio $p_{i+1}(t)/p_i(t)$ is a non-decreasing function of time.

The concept of weighted distributions is important in a wide range of statistical applications. A density function g is said to be a weighted density function corresponding to density function f with weight $w>0$ if

$$g(x)=\frac{w(x)f(x)}{E(w(X))}, \quad x\in R,$$

where $E(w(X))<\infty$. The following theorem shows that $p_i(t)$ is a weighted distribution corresponding to $q_i(t)$ $(r_i(t))$. Before giving the theorem, we recall that the hazard rate (reversed hazard rate) of a continuous random variable X with density $f(t)$, distribution F, and survival function $\bar{F}(t)$, which we denote by $\lambda(t)$ $(h(t))$, is defined as $\lambda(t)=f(t)/\bar{F}(t)$ $(h(t)=f(t)/F(t))$.

Theorem 2.3.

(a) *Let $\lambda_{i:n}(t)$ denote the hazard rate of $X_{i:n}$. For $i=1,2,\ldots,n$, we have*

$$p_i(t)=\frac{\lambda_{i:n}(t)}{E\left(\lambda_{Y_1:n}(t)\right)}q_i(t), \quad t>0,$$

where the expectation is taken over the discrete random variable Y_1 with probability function $q_i(t)$.

(b) *Let $h_{i:n}(t)$ denote the reversed hazard rate of $X_{i:n}$. For $i=1,2,\ldots,n$, we have*

$$p_i(t)=\frac{h_{i:n}(t)}{E\left(h_{Y_2:n}(t)\right)}r_i(t), \quad t>0,$$

where the expectation is taken over the discrete random variable Y_2 with probability function $r_i(t)$.

Proof: We only prove part (a), as (b) can be proven similarly. Using (3), and (6), we have

$$\begin{aligned}\frac{p_i(t)}{q_i(t)} &= \frac{f_{i:n}(t)}{\bar{F}_{i:n}(t)}\frac{\sum_{j=1}^n s_j\bar{F}_{j:n}(t)}{\sum_{j=1}^n s_j f_{j:n}(t)}\\ &= \frac{\lambda_{i:n}(t)}{\lambda_T(t)}\end{aligned}$$

where $\lambda_T(t)$ denotes the hazard rate of the system. But we have

$$\begin{aligned}\lambda_T(t) &= \frac{\sum_{j=1}^n s_j f_{j:n}(t)}{\sum_{j=1}^n s_j\bar{F}_{j:n}(t)}\\ &= \frac{\sum_{j=1}^n s_j\lambda_{i:n}(t)\bar{F}_{j:n}(t)}{\sum_{j=1}^n s_j\bar{F}_{j:n}(t)}\\ &= E\left(\lambda_{Y_1:n}(t)\right).\end{aligned}$$

□

The following lemma enables us to prove the main results of this paper.

Lemma 2.4. *The functions $\eta_{1j}(t)$, $\eta_{2j}(t)$, and $\eta_j(t)$, $j = 1, 2, \ldots, n$, defined as follows, respectively, are non-decreasing functions of t, $t > 0$.*

(a) $\eta_{1j}(t) = \frac{\sum_{k=j}^{n} s_k \sum_{m=0}^{k-1} \binom{n}{m} t^m}{\sum_{i=1}^{n} s_i \sum_{m=0}^{i-1} \binom{n}{m} t^m}$,

(b) $\eta_{2j}(t) = \frac{\sum_{k=j}^{n} s_k \sum_{m=k}^{n} \binom{n}{m} t^m}{\sum_{i=1}^{n} s_i \sum_{m=i}^{n} \binom{n}{m} t^m}$, *and*

(c) $\eta_j(t) = \frac{\sum_{k=j}^{n} s_k c_{k,n} t^k}{\sum_{i=1}^{n} s_i c_{i,n} t^i}$.

Proof: We first prove (a). Navarro et al. [10], showed that, for all $j = 1, 2, \ldots, n$, the survival function $\bar{Q}_j(t) = \sum_{k=j}^{n} q_k(t)$, is a non-decreasing function of t. In particular, if we assume that $\phi(t) = t$, i.e. $F(t) = t/t+1$, $t > 0$, then $\eta_{1j}(t) = \bar{Q}_j(t)$ is a non-decreasing function of t. This completes the proof of part (a).

(b) Using the same steps as used in Lemma 2.1 of Navarro et al. [10] to show that $\bar{Q}_j(t)$ is non-decreasing in t, and based on the fact that the ratio $F_{i:n}(t)$ $F_{i+1:n}(t)$ is a non-increasing function of t, we can show that, for all $j = 1, 2, ..., n$, $\bar{R}_j(t) = \sum_{k=j}^{n} r_k(t)$ is a non-decreasing function of t. Thus, by taking $\phi(t) = t$, we can easily see that $\eta_{2j}(t)$ is a non-decreasing function of t. Part (c) can be considered as a special case of either part (a) or part (b). Hence, the proof of the lemma is complete. □

Now we can prove the following interesting result.

Theorem 2.5. *Consider two coherent systems of order n with the same structures. Assume that the components of the systems have independent lifetimes, with survival functions $\bar{F}$, and $\bar{G}$, respectively. If for all t, $\bar{F}(t) \leq \bar{G}(t)$, then*

(a) $\bar{Q}_j^F(t) \geq \bar{Q}_j^G(t)$, *where* $\bar{Q}_j^F(t) = \sum_{k=j}^{n} q_k^F(t)$, $j = 1, 2, \ldots, n$, *and* $\bar{Q}_j^G(t)$ *is defined similarly;*

(b) $\bar{R}_j^F(t) \geq \bar{R}_j^G(t)$, *where* $\bar{R}_j^F(t) = \sum_{k=j}^{n} r_k^F(t)$, $j = 1, 2, \ldots, n$, *and* $\bar{R}_j^G(t)$ *is defined similarly; and*

(c) $\bar{P}_j^F(t) \geq \bar{P}_j^G(t)$, *where* $\bar{P}_j^F(t) = \sum_{k=j}^{n} p_k^F(t)$, $j = 1, 2, \ldots, n$, *and* $\bar{P}_j^G(t)$ *is defined similarly.*

Proof: We prove part (a). Parts (b) and (c) can be proven similarly. The assumption that $\bar{F}(t) \leq \bar{G}(t)$ implies that $\phi_F(t) \geq \phi_G(t)$, $t > 0$. Hence, from part (a) of Lemma 2.4, we have

$$\eta_{1j}(\phi_F(t)) \geq \eta_{1j}(\phi_G(t)),$$

which is equivalent to saying that, for all $j = 1, 2, ..., n$, and all t, $\bar{Q}_j^F(t) \geq \bar{Q}_j^G(t)$. This completes the proof of the theorem. □ A simple, interesting conclusion of the theorem is as follows. Assume that the signature of the system is given as (4). From Theorem 2.5, it can be concluded that, when $\bar{F}(t) \leq \bar{G}(t)$, then

$$P(T = X_{j:n} | T \in A) \geq P(T = Y_{j:n} | T \in A)$$

when $X_{j:n}$, and $Y_{j:n}$ are the lifetimes of the jth ordered lifetime in the systems based on F, and G respectively; and A is the set $A = \{t\}$, $A = (t, \infty)$ or $A = (0, t)$. That is, under these circumstances, in the system that has less reliable components, it is more probable that the component with the jth ordered lifetime causes the failure of the system. Denote by $E(t)$, $E_1(t)$, and $E_2(t)$ the expectations of Y, Y_1, and Y_2, respectively. Then we can prove the following representation theorem.

Theorem 2.6. *The survival function of T, $\bar{F}_T(t)$, can be represented as*

$$\bar{F}_T(t) = \exp\{-\int_0^t \frac{\frac{d}{dx} E_1(x)}{E_1(x) - E(x)} dx\}, \quad t > 0. \tag{10}$$

Proof: Under the assumption that $q_i(t)$, $i = 1, 2, ..., n$, is differentiable in terms of t, we have, for $i = 1, 2, \ldots, n$,

$$\begin{aligned} \frac{d}{dx} q_i(x) &= \frac{-s_i f_{i:n}(x) \sum_{j=1}^n s_j \bar{F}_{j:n}(x) + s_i \bar{F}_{i:n}(x) \sum_{j=1}^n s_j f_{j:n}(x)}{\left(\sum_{i=1}^n s_j \bar{F}_{j:n}(x)\right)^2} \\ &= r_T(t) \left(q_i(t) - p_i(t)\right), \end{aligned} \tag{11}$$

where $r_T(t) = \frac{\sum_{i=1}^n s_i f_{i:n}(t)}{\sum_{i=1}^n s_i \bar{F}_{i:n}(t)}$ denotes the hazard rate of the system. On the other hand, we have

$$\begin{aligned} \frac{d}{dx} E_1(x) &= \sum_{i=1}^n x_{i:n} \frac{d}{dx} q_i(x) \\ &= r_T(x) \sum_{i=1}^n x_{i:n} \left(q_i(t) - p_i(t)\right) \\ &= r_T(x) \left(E_1(x) - E(x)\right), \end{aligned}$$

where the second equality follows from (11). Using the fact that $r_T(t) = \frac{f_T(t)}{\bar{F}_T(t)}$, where $f_T(t)$ is the density function of the system, we obtain (10). □ Using the same argument as used to prove Theorem 2.6, it can be shown that the distribution function of T, $F_T(t)$, can be recovered from $E(t)$, and $E_2(t)$.

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Distribution of Order Statistics for Exchangeable Random Variables

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Abstract

Let $T_1, T_2, \ldots, T_n$ be exchangeable random variables and have an absolutely continuous joint distribution and suppose $T_{i:n}$ represents the ith order statistic among T_i's, $i = 1, 2, \ldots, n$. In this paper some expressions for the joint distribution of $(T_{1:n}, \ldots, T_{n:n})$, marginal distribution of $T_{i:n}$ and the joint distribution of $(T_{r:n}, T_{k:n})$, $1 \leq r \leq k \leq n$ in terms of the joint distribution (or joint survival) function of T_i's are provided, which are more general than those of the case when T_i's are independent and identically distributed (i.i.d.) random variables. Using these and when $\{T_1, \ldots, T_n\}$ is a sequence of lifetimes, some expressions for the mean residual life functions of a $n-k+1$-out-of-n system, $H_n^k(t) = E(T_{k:n} - t | T_{1:n} > t)$ and $M_n^{r,k}(t) = E(T_{k:n} - t | T_{r:n} > t)$, $1 \leq r \leq k \leq n$ in terms of the joint survival function of T_i's are given. Also some examples are provided.

Keywords and Phrases: Order statistics; Exchangeable random variables; $(n-k+1)$-out-of-n system; Mean residual life function.

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1 Introduction

In reliability analysis, the assumption of dependence among lifetimes of components of the system is more realistic than assumption of independence. For example the components of the system may be affected by a common shock, see e.g. Barlow and Proschan (1975). A kind of dependence is exchangeability which attracted the interest of many authors in recent years. See for example Navarro et al. (2005), Navarro et al. (2007) and Zhang (2010).

The random variables $T_1, T_2, \ldots, T_n$ are exchangeable if

$$P(T_1 \leq t_1, \ldots, T_n \leq t_n) = P(T_{\pi(1)} \leq t_1, \ldots, T_{\pi(n)} \leq t_n)$$

where $\pi = (\pi(1), \ldots, \pi(n))$ is an arbitrary permutation of $\{1, \ldots, n\}$, i.e. the joint distribution of $T_1, \ldots, T_n$ is symmetric in $t_1, \ldots, t_n$. Note that T_i's are identically distributed. We assume that all random variables here are absolutely continuous. It is well known that when $T_1, \ldots, T_n$ are independent and have a common distribution function F, survival function $\bar{F} = 1 - F$ and density function $f = F'$ then

$$f_{(T_{1:n}, \ldots, T_{n:n})}(t_1, \ldots, t_n) = n! f_{(T_1, \ldots, T_n)}(t_1, \ldots, t_n) = n! \prod_{i=1}^{n} f(t_i),$$

$$\bar{F}_{T_{i:n}}(t) = P(T_{i:n} > t) = \sum_{j=0}^{i-1} \binom{n}{j} F^j(t) \bar{F}^{n-j}(t),$$

$$f_{T_{i:n}}(t) = \frac{n!}{(i-1)!(n-i)!} F^{i-1}(t) \bar{F}^{n-i}(t) f(t)$$

and

$$P(T_{r:n} > t, T_{k:n} > s) = \sum_{i=0}^{r-1} \binom{n}{i} F^i(t) \sum_{j=0}^{k-i-1} \binom{n-i}{j} [\bar{F}(t) - \bar{F}(s)]^j \bar{F}^{n-i-j}(s)$$

for $s > t$ and $1 \leq r \leq k \leq n$, see for example David and Nagaraja (2003). In Section 2 some expressions for the right sides of the above formulas are given when $T_1, \ldots, T_n$ are exchangeable random variables. Finally in Section 3, some explicit formulas for the mean residual life (MRL) functions $H_n^k(t) = E(T_{k:n} - t | T_{1:n} > t)$ and $M_n^{r,k}(T) = E(T_{k:n} - t | T_{r:n} > t)$ of a $n-k+1$-out-of-n system with exchangeable components in terms of the joint survival function of T_i's, $\bar{\mathbf{F}}(t_1, \dots, t_n) = P(T_1 > t_1, \dots, T_n > t_n)$ are provided.

2 Main Results

Let $(T_1, T_2, \dots, T_n)$ be an exchangeable random vector, that is $P(T_1 \le t_1, \dots, T_n \le t_n) = P(T_{\pi(1)} \le t_1, \dots, T_{\pi(n)} \le t_n)$, for any permutation $\pi = (\pi(1), \dots, \pi(n))$ of $\{1, \dots, n\}$ and suppose $T_{1:n}, \dots, T_{n:n}$ are corresponding order statistics. We note that the joint density function $f_{(T_1,\dots,T_n)}(t_1, \dots, t_n)$ is also symmetric in $t_1, \dots, t_n$. Therefore we can write

$$f_{(T_{1:n},\dots,T_{n:n})}(x_1, \dots, x_n) = n! f_{(T_1,\dots,T_n)}(x_1, \dots, x_n), \quad x_1 < x_2 < \dots < x_n. \tag{1}$$

We now consider the survival function of $T_{i:n}$. Since T_i's are exchangeable random variables we can write

$$\bar{F}_{T_{i:n}}(t) = P(T_{i:n} > t) = \sum_{j=0}^{i-1} \binom{n}{j} P(T_1 \le t, \dots, T_j \le t, T_{j+1} > t, \dots, T_n > t).$$

The above equation can be written in terms of the joint survival function of T_i's which is given as follow.

Lemma 1. We have

$$\bar{F}_{T_{i:n}}(t) = \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} P(T_{1:j} > t)$$

$$= \sum_{j=n-i+1}^{n} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n}{j} \bar{\mathbf{F}}(\underbrace{t, \dots, t}_{j}, \underbrace{-\infty, \dots, -\infty}_{n-j})$$

$$= 1 - \sum_{j=i}^{n} (-1)^{j-i} \binom{j-1}{i-1} \binom{n}{j} \mathbf{F}(\underbrace{t, \dots, t}_{j}, \underbrace{\infty, \dots, \infty}_{n-j}) = 1 - F_{T_{i:n}}(t). \tag{2}$$

Proof. Using Equation (3.4.2) in David and Nagaraja (2003, Page 46) for i.i.d. random variables which exactly holds true for exchangeable random variables the proof follows.

Equation (2) shows that the survival (or distribution) function of $T_{i:n}$ can be written as a linear combination of the joint (or survival) function of $T_1, \dots, T_n$. By taking derivative from both sides of the Equation (2) with respect to t, density function of $T_{i:n}$ can be simply obtained.

We now consider the bivariate distribution of $(T_{r:n}, T_{k:n})$, $1 \le r < k \le n$. For $r = 1$ and $s > t$ we have

$$P(T_{1:n} > t, T_{k:n} > s) = \sum_{j=0}^{k-1} \binom{n}{j} P(t < T_1 \le s, \dots, t < T_j \le s, T_{j+1} > s, \dots, T_n > s).$$

Lemma 2. For $s > t$ and $1 \le k \le n$ we have $\bar{F}_{(T_{1:n}, T_{k:n})}(t, s) = P(T_{1:n} > t, T_{k:n} > s)$

$$= \sum_{j=0}^{k-1} (-1)^{k-j-1} \binom{n}{j} \binom{n-j-1}{n-k} P(T_1 > t, \dots, T_j > t, T_{j+1} > s, \dots, T_n > s)$$

$$= \sum_{j=0}^{k-1} (-1)^{k-j-1} \binom{n}{j} \binom{n-j-1}{n-k} \bar{\mathbf{F}}(\underbrace{t, \dots, t}_{j}, \underbrace{s, \dots, s}_{n-j})$$

$$= \sum_{j=n-k+1}^{n} (-1)^{j-n+k-1} \binom{n}{j} \binom{j-1}{n-k} \bar{\mathbf{F}}(\underbrace{t, \dots, t}_{n-j}, \underbrace{s, \dots, s}_{j}) \tag{3}$$

Proof. The proof follows from Equation (3.4.3) in David and Nagaraja (2003, Page 46). Equation (3) shows that the joint reliability of $(T_{1:n}, T_{k:n})$ can be written as a linear combination of the joint survival function of T_i's, $\bar{\mathbf{F}}(x_1, \ldots, x_n) = P(T_1 > x_1, \ldots, T_n > x_n)$.

Note that by using the joint and marginal reliability functions, the joint distribution function of $(T_{1:n}, T_{k:n})$ can be simply obtained as follow:

$$F_{(T_{1:n},T_{k:n})}(t,s) = P(T_{1:n} \le t, T_{k:n} \le s) = 1 - \bar{F}_{T_{1:n}}(t) - \bar{F}_{T_{k:n}}(s) + \bar{F}_{(T_{1:n},T_{k:n})}(t,s).$$

We now consider the joint reliability function of $(T_{r:n}, T_{k:n})$ when $1 < r < k \le n$. By using the corresponding properties for order statistics and exchangeability assumption of T_i's we can write

$$P(T_{r:n} > t, T_{k:n} > s) = \sum_{i=0}^{r-1} \binom{n}{i} \sum_{j=0}^{k-i-1} \binom{n-i}{j} \times$$

$$P(T_1 \le t, \ldots, T_i \le t, t < T_{i+1} \le s, \ldots, t < T_{i+j} \le s, T_{i+j+1} > s, \ldots, T_n > s). \tag{4}$$

Lemma 3. We have

$$P(T_1 \le t, \ldots, T_i \le t, t < T_{i+1} \le s, \ldots, t < T_{i+j} \le s, T_{i+j+1} > s, \ldots, T_n > s)$$

$$= \bar{\mathbf{F}}(\underbrace{-\infty,\ldots,-\infty}_{i},\underbrace{t,\ldots,t}_{j},\underbrace{s,\ldots,s}_{n-i-j}) - \sum_{l=1}^{j}(-1)^{l+1}\binom{j}{l}\bar{\mathbf{F}}(\underbrace{-\infty,\ldots,-\infty}_{i},\underbrace{t,\ldots,t}_{j-l},\underbrace{s,\ldots,s}_{n-i-j+l})$$

$$-\sum_{l=1}^{i}(-1)^{l+1}\binom{i}{l}\bar{\mathbf{F}}(\underbrace{-\infty,\ldots,-\infty}_{i-l},\underbrace{t,\ldots,t}_{l+j},\underbrace{s,\ldots,s}_{n-i-j})$$

$$+\sum_{l_1=1}^{i}\sum_{l_2=1}^{j}(-1)^{l_1+l_2}\binom{i}{l_1}\binom{j}{l_2}\bar{\mathbf{F}}(\underbrace{-\infty,\ldots,-\infty}_{i-l_1},\underbrace{t,\ldots,t}_{j+l_1-l_2},\underbrace{s,\ldots,s}_{n-i-j+l_2}). \tag{5}$$

We assume that $\sum_{l=1}^{N} a_l = 0$ when $N = 0$.

Proof. We can write

$$P(T_1 \le t, \ldots, T_i \le t, t < T_{i+1} \le s, \ldots, t < T_{i+j} \le s, T_{i+j+1} > s, \ldots, T_n > s) = P(A \cap B' \cap C')$$

where the events A, B and C are defined as follows

$$A = (T_{i+1} > t, \ldots, T_{i+j} > t, T_{i+j+1} > s, \ldots, T_n > s), B = \cup_{l=i+1}^{i+j}(T_l > s), C = \cup_{l=1}^{i}(T_l > t).$$

Using the principle of inclusion-exclusion and noting that $P(A \cap B' \cap C') = P(A) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$ the result follows.

If we replace Equation (5) in Equation (4), we can write the joint reliability function of $(T_{r:n}, T_{k:n})$, again as a linear combination of the joint survival function of T_i's.

3 Function of a $n-k+1$-out-of-n System with Exchangeable Components

The mean residual life (MRL) and the failure rate functions are very important in Reliability analysis. In this section we assume that the exchangeable and nonnegative random variables $T_1, \ldots, T_n$ represent the lifetimes of n components which are connected in a $n-k+1$-out-of-n system. It is well known that the lifetime of this system is $T_{k:n}$. From the results given in the previous section, we shall obtain the MRL function of the system. Particularly we consider two MRL functions

$$H_n^k(t) = E(T_{k:n} - t | T_{1:n} > t) \text{ and } M_n^{r,k}(t) = E(T_{k:n} - t | T_{r:n} > t),\ 1 \le r \le k \le n,$$

in which $H_n^k(t)$ measures MRL of the system when all components of the system are working at or before time t whereas in $M_n^{r,k}(t)$ the number of those components is at least $n-r+1$.

The following lemma gives an expression for $H_n^k(t)$.

Lemma 4. We have

$$H_n^k(t) = E(T_{k:n} - t | T_{1:n} > t) = \sum_{j=n-k+1}^{n} (-1)^{j-n+k-1} \binom{n}{j} \binom{j-1}{n-k} H_{j,n}^1(t) \tag{6}$$

where

$$H_{j,n}^1(t) = E(T_{1:j} - t | T_{1:n} > t) = \int_0^\infty \frac{\bar{\mathbf{F}}(\overbrace{t+x, \ldots, t+x}^{j}, \overbrace{t, \ldots, t}^{n-j})}{\bar{\mathbf{F}}(t, \ldots, t)} dx.$$

Proof. We know that

$$H_n^k(t) = E(T_{k:n} - t | T_{1:n} > t) = \int_0^\infty P(T_{k:n} - t > x | T_{1:n} > t) dx = \int_0^\infty \frac{P(T_{k:n} > t+x, T_{1:n} > t)}{\bar{\mathbf{F}}(t, \ldots, t)} dx.$$

Now from Equation (3) the proof follows.

Equation (6) gives an expression for $H_n^k(t) = E(T_{k:n} - t | T_{1:n} > t)$ in terms of the joint survival function of T_i's, $\bar{\mathbf{F}}(x_1, \ldots, x_n)$.

Remark. We note that from Lemma 1 and in view of Lemma 3, the MRL function

$$M_n^{r,k}(t) = E(T_{k:n} - t | T_{1:n} > t) = \int_0^\infty \frac{P(T_{k:n} > t+x, T_{r:n} > t)}{P(T_{r:n} > t)} dx$$

can also be written in terms of the joint survival function of T_i's but the expression becomes lengthy.

We now give two examples for determining $H_n^k(t)$.

Example 1. Suppose that the joint distribution of $T_1, \ldots, T_n$ is Marshal and Olkin's multivariate exponential with the following survival function

$$\bar{\mathbf{F}}(x_1, \ldots, x_n) = exp\{-\sum_{i=1}^{n} \lambda_i x_i - \sum_{i_1 < i_2} \lambda_{i_1,i_2} \max(x_{i_1}, x_{i_2}) - \cdots - \lambda_{12\ldots n} \max(x_1, \ldots, x_n)\}.$$

For the special case $\lambda_1 = \cdots = \lambda_n = \lambda_{12} = \cdots = \lambda_{12\ldots n} = \lambda$, $\bar{\mathbf{F}}(x_1, \ldots, x_n)$ is exchangeable. It can be simply shown that

$$\frac{\bar{\mathbf{F}}(\overbrace{t+x, \ldots, t+x}^{j}, \overbrace{t, \ldots, t}^{n-j})}{\bar{\mathbf{F}}(t, \ldots, t)} = exp\{-(2^n - 2^{n-j})\lambda x\}$$

and hence from Equation (6) we have

$$H_n^k(t) = \lambda^{-1} \sum_{i=n-k+1}^{n} \frac{(-1)^{i+k-n-1}}{2^n - 2^{n-i}} \binom{n}{i} \binom{i-1}{n-k} = \lambda^{-1} \sum_{i=0}^{k-1} \frac{(-1)^{k-i-1}}{2^n - 2^i} \binom{n}{i} \binom{n-i-1}{n-k},$$

which is a positive constant. Another special case corresponds to $\lambda_1 = \cdots = \lambda_n = \lambda > 0$, and other λ's equal to 0 (i.e., the i.i.d. case). In this case we have

$$\frac{\bar{\mathbf{F}}(\overbrace{t+x, \ldots, t+x}^{j}, \overbrace{t, \ldots, t}^{n-j})}{\bar{\mathbf{F}}(t, \ldots, t)} = exp(-\lambda j x)$$

and therefore

$$H_n^k(t) = \lambda^{-1} \sum_{i=n-k+1}^{n} \frac{(-1)^{i+k-n-1}}{i} \binom{n}{i} \binom{i-1}{n-k} = \lambda^{-1} \sum_{i=0}^{k-1} \frac{(-1)^{k-i-1}}{n-i} \binom{n}{i} \binom{n-i-1}{n-k},$$

which is again a positive constant. In Example 1 we note that $P(T_1 = T_2 = \cdots = T_n) > 0$, because T_i's are subjected to a common shock and therefore their joint distribution is not absolutely continuous. See the following example.

Example 2. Assume that $T_1, \ldots, T_n$ are distributed as Mardia's multivariate Pareto distribution with the following joint survival function

$$\bar{\mathbf{F}}(x_1, \ldots, x_n) = (\theta^{-1} \sum_{i=1}^{n} x_i - n + 1)^{-a}, x_i > \theta > 0, a > 1.$$

It can be simply shown that

$$E(T_{1:j} - t | T_{1:n} > t) = \int_0^\infty \frac{\bar{\mathbf{F}}(\overbrace{t + x, \ldots, t + x}^{j}, \overbrace{t, \ldots, t}^{n-j})}{\bar{\mathbf{F}}(t, \ldots, t)} dx = \frac{nt - n\theta + \theta}{j(a-1)}$$

and hence from Equation (6), for $t > \theta$ we have

$$H_n^k(t) = \sum_{j=n-k+1}^{n} (-1)^{j+k-n-1} \binom{n}{j} \binom{j-1}{n-k} \frac{nt - n\theta + \theta}{j(a-1)} = \left(\sum_{j=n-k+1}^{n} c_j(k, n) \right) \frac{nt - n\theta + \theta}{a-1}$$

where $c_j(k, n) = (-1)^{j+k-n-1} \binom{n}{j} \binom{j-1}{n-k} / j$. Note that $\sum_{j=n-k+1}^{n} c_j(k, n) = (a-1)/\theta H_n^k(\theta) \geq 0$ and therefore $H_n^k(t)$ is a linearly increasing function of t.

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On the Residual Lifelengths of the Remaining Components in a Coherent System

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Abstract

In this note, we consider a coherent system with the property that, upon failure of the system, some of its components remain unfail in the system. Under this condition, we study the residual lifetime of the live components of the system. Signature based mixture representation of the joint and marginal reliability functions of the live components are obtained. Various stochastic and aging properties of the residual lifetime of such components are investigated. Some characterization results on exponential distributions are also provided.

Keywords and Phrases: Order statistics, stochastic order, residual lifetime, mixture of distributions, signature, hazard rate, mean residual life.

AMS Subject Classification 2000: primary 62N05; secondary 62G30.

1 Introduction

Consider a technical system consisting of n components with structure function ϕ. The system is said to be a coherent system if ϕ is monotone in every component (that is, the state of ϕ can not become worse if any of its elements changes its state from down to up) and the system has no irrelevant component (a component is said to be irrelevant if its performance does not affect the performance of the system). In recent years, various aspects of reliability and aging characteristics of the lifetime and the residual lifetime of coherent systems and its components have been studied by different researchers. The concept of the *signature* of the system has shown to be very useful tool in investigating the reliability specifications of the system. Suppose that a coherent system with n components has independent and identically distributed (i.i.d.) lifetimes $X_1, X_2, ..., X_n$, where X_i's are distributed according to a common continuous distribution F. Let $T = T(X_1, ..., X_n)$ denote the lifetime of the system. Then, the signature of a coherent system is defined to be a probability vector $\mathbf{s} = (s_1, s_2, ..., s_n)$ in which

$$s_i = P\{T = X_{i:n}\}, \quad i = 1, 2, ..., n, \tag{1}$$

where $X_{i:n}$ represents the ith ordered components lifetime. It can be shown that $s_i = \frac{n_i}{n!}$, where n_i is the number of ways that distinct $X_1, X_2, \ldots, X_n$ can be ordered such that $T(x_1, ..., x_n) = x_{i:n}$, $i = 1, ..., n$, where $x_{i:n}$ is the value of $X_{i:n}$ (see, Samaniego [3]). The probability s_i is, in fact, the probability that the component with lifetime $X_{i:n}$ causes the failure of the system. Based on the fact that the signature vector $\mathbf{s}$ does not depend on the common distribution function of X_i's, the survival function of T, can be written as a mixture of the survival functions of $X_{i:n}$, $i = 1, 2, ..., n$, with weights $s_1, s_2, ..., s_n$. That is, if $\bar{F}_T(t)$ denotes the survival function of the system lifetime and $\bar{F}_{i:n}(t)$ denotes the survival function of $X_{i:n}$, then

$$\bar{F}_T(t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t). \tag{2}$$

Representation (2) is a useful identity to study the stochastic properties of a coherent system based on the properties of its signatures as well as the properties of ordered lifetime of its components. Kochar et al. [?] showed that when the vector of signatures of two coherent systems are stochastically ordered then the lifetimes of the systems are also stochastically ordered.

In a recent paper, Biramov and Arnold [1], considered an $(n-k+1)$-out-of-n system and studied the properties of the residual life lengths of the remaining functioning components of the system. Since the signature vector of $(n-k+1)$-out-of-n systems is of the form

$$\mathbf{s} = (\underbrace{0, 0..., 0,}_{k-1 \text{ times}}\ 1,\ \underbrace{0, 0..., 0}_{n-k \text{ times}})$$

the system has the property that, upon its failure, $n-k$ of its components will remain unfail and hence, after they rescue from the failed system, they can be possibly used in other systems. If this is so, then an interesting problem would be the study on the reliability properties of those live components. Suppose that $X_1, ..., X_n$ represent the lifetime of the components of a $(n-k+1)$-out-of-n system which are i.i.d. with common continuous distribution function F and density function f. Denote by $X_1^{(k)}, ..., X_{n-k}^{(k)}$, the residual lifetime of the components after the k failures in the system. Bairamov and Arnold have investigated several distributional properties of $X_j^{(k)}$'s $j = 1, ..., n-k$. Among other results, they showed that the joint survival function of $X_j^{(k)}$'s is given as

$$\bar{F}_n^{(k)}(x_1, ..., x_{n-k}) = \int_0^\infty \left\{ \prod_{j=1}^{n-k} \frac{\bar{F}(t+x_j)}{\bar{F}(t)} \right\} dF_{k:n}(t), \tag{3}$$

where $F_{k:n}$ denotes the distribution of the kth order statistics $X_{k:n}$. In this paper, we extend the work of Bairamov and Arnold to the case where the system is a coherent system. More precisely, we consider a coherent system for which the signature vector is of the form $\mathbf{s} = (s_1, ..., s_i, 0, ..., 0)$ where $s_j > 0$, $j = 1, ..., i$. It is clear that, a coherent system with this vector of signature has an $(n-k+1)$-out-of-n system as a special case. Since $s_j = 0$ for $j = i+1, ..., n$, then after system failure, at least $n-i$ components remain alive and hence they possibly can be separated from the system and used in the other systems. The aim of the present paper is to investigate distributional properties of the residual lifetime of unfail components of the system.

2 Main Results

Consider a coherent system with a signature vector of the form

$$\mathbf{s} = (s_1, ..., s_i, 0, ..., 0) \tag{4}$$

where $s_j > 0$, $j = 1, ..., i$. Suppose that $X_1, X_2, ..., X_n$ denote the components lifetime of the system where X_i's are assumed to be i.i.d. with a common absolutely continuous distribution function F and density function f. If $X_{1:n}, X_{2:n}, ..., X_{n:n}$ represent the ordered lifetimes of the components, then in a system with signature vector (4), the components with lifetime $X_{i+1:n}, ..., X_{n:n}$ would never cause the failure of the system. Hence, after the failure of the system these components remain unfail. Denote by $Y_j^{(i)}$, $j = 1, ..., n-i$, the randomly ordered values of $X_{j:n}$, $j = i+1, ..., n$. Then, the residual lifetime of the live components after the failure of the system can be denoted by

$$X_j^* = Y_j^{(i)} - T, \quad j = 1, ..., n-i,$$

where T represents the lifetime of the system. The joint survival function of the X_j^*'s can be expressed as

$$\bar{F}^*(x_1, ..., x_{n-i}) = \int_0^\infty \left\{ \prod_{j=1}^{n-i} \frac{\bar{F}(t+x_j)}{\bar{F}(t)} \right\} f_T(t)dt, \tag{5}$$

where $\bar{F}(t) = 1 - F(t)$ and $f_T(t)$ denotes the density function of T. It is seen from (5) that when T is given, X_j^*'s are independent, i.e. they are exchangeable.

Also from (3) and (5), $\bar{F}^*(x_1,...,x_{n-i})$ can be represented as a mixture distribution of the survival probabilities of unfailed components in $(n-k+1)$-out-of-n systems, as follows

$$\begin{aligned}\bar{F}^*(x_1,...,x_{n-i}) &= \sum_{k=1}^{i} s_k \int_0^\infty \left\{\prod_{j=1}^{n-i} \frac{\bar{F}(t+x_j)}{\bar{F}(t)}\right\} dF_{k:n}(t) \\ &= \sum_{k=1}^{i} s_k \bar{F}_n^{(k)}(x_1,...,x_{n-k}).\end{aligned}$$

The common marginal survival function and marginal density functions of X_j^*'s are, respectively, given as

$$\bar{F}^*(x) = \int_0^\infty \frac{\bar{F}(t+x)}{\bar{F}(t)} f_T(t)dt \tag{6}$$

and

$$f^*(x) = \int_0^\infty \frac{f(t+x)}{\bar{F}(t)} f_T(t)dt. \tag{7}$$

Note that (6) can be rewritten as

$$\begin{aligned}\bar{F}^*(x) &= \sum_{k=1}^{i} s_k \int_0^\infty \frac{\bar{F}(t+x)}{\bar{F}(t)} dF_{k:n}(t) \\ &= \sum_{k=1}^{i} s_k . \bar{F}_n^{(k)}(x),\end{aligned} \tag{8}$$

where, $\bar{F}_n^{(k)}(x)$ is obtained from (3). That is, the survival function $\bar{F}^*(x)$ can be represented as a mixture of survival functions $\bar{F}_n^{(k)}(x)$.
If $r^*(t)$ denotes the corresponding hazard rate of $\bar{F}^*$, then it can be shown that

$$r^*(x) = \frac{f^*(x)}{\bar{F}^*(x)} = \int_0^\infty r(x+t)dG_x(t)$$

where $r(t)$ is the hazard rate of $\bar{F}$ and

$$dG_x(t) = \frac{\frac{\bar{F}(t+x)}{\bar{F}(t)} f_T(t)dt}{\int_0^\infty \frac{\bar{F}(u+x)}{\bar{F}(u)} f_T(u)du}.$$

A simple conclusion of this representation is that, if the components lifetime of the system are IFR (DFR) [1], then

$$\begin{aligned}r^*(x) &= \int_0^\infty r(x+t)dG_x(t) \\ &\geq (\leq) \; r(x).\end{aligned}$$

Similarly, if $m^*(t)$ and $m(t)$ denote the MRL [2] associated to $\bar{F}^*$, and $\bar{F}$, respectively, then it is easily seen that

$$m^*(x) = \int_0^\infty m(x+t)dG_x(t)$$

This, in turn, yields to the following fact that if F is DMRL (IMRL) [3], then for $t > 0$,

$$m^*(t) \leq (\geq) m(t).$$

[1]Increasing Failure Rate (Decreasing Failure Rate)
[2]Mean Residual Life
[3]Decreasing Mean Residual Life (Increasing Mean Residual Life)

2.1 A link with mean residual life

The MRL of a lifetime random variable T with survival function $\bar{F}$, denoted by $m_F(t)$ is defined as

$$m_F(t) = E(X - t|X > t) = \frac{\int_t^\infty \bar{F}(x)}{\bar{F}(t)}dx$$

For the aforementioned system,

$$E(X_1^*) = E(m_F(T)). \tag{9}$$

A simple consequence of Theorem 2.1 is that, if F is NBUE (NWUE)[4] with mean μ, then

$$E(X_1^*) \leq (\geq)\mu.$$

Example *Consider a series-parallel system with three components with lifetime*

$$T = \min(X_1, \max(X_2, X_3)),$$

where X_i is the lifetime of ith component. The signature vector of such system is equal to $s = (\frac{1}{3}, \frac{2}{3}, 0)$. Therefore, (8) is written as

$$\bar{F}^*(x) \quad = \quad \frac{1}{3}\int_0^\infty \frac{\bar{F}(t+x)}{\bar{F}(t)}dF_{1:3}(t) + \frac{2}{3}\int_0^\infty \frac{\bar{F}(t+x)}{\bar{F}(t)}dF_{2:3}(t),$$

Now, assume that X has a GPD distribution with survival function

$$\bar{F}(x) = \left(\frac{b}{ax+b}\right)^{\frac{1}{a}+1}, \qquad x > 0,\ a > -1,\ b > 0.$$

The GPD is a family of distributions which includes, the exponential distribution (when $a \to 0$), the Pareto distribution (for $a > 0$) and the Power distribution (for $-1 < a < 0$). For such distribution it is easily seen that $m_F(x) = ax + b$. Hence from (9), we have

$$E(X_1^*) = E(aT + b) = aE(T) + b,$$

where T is the lifetime of the system. To compute the expectation of T, one can easily show that the survival function of the system is given as follows.

$$\bar{F}_T(t) = \bar{F}(t)[1 - F^2(t)] = \bar{F}^2(t)[2 - \bar{F}(t)], \quad t > 0.$$

Hence

$$E(T) \quad = \quad \int_0^\infty \bar{F}_T(t)dt = \frac{b(3a+4)}{(3+2a)(2+a)}.$$

Therefore

$$E(X_1^*) = \frac{b(5a^2 + 11a + 6)}{(3+2a)(2+a)}.$$

Note that, for GPD, $E(X) = b$. Hence we have

$$E(X_1^*) \begin{cases} > E(X) & \text{for Pareto distribution} \\ = E(X) & \text{for exponential distribution} \\ < E(X) & \text{for power distribution.} \end{cases}$$

[4] New Better than Used in Expectation (New Worse than Used in Expectation)

2.2 Characterization of exponential distribution

Assuming that the component lifetime distribution F was an exponential distribution, the residual lifetimes following the system failure will be independent and will have the same marginal distribution as that of the original lifetimes. $X_1^* \stackrel{d}{=} X_1$ if and only if $X_1 \sim Exponential(\lambda)$ for some $\lambda > 0$. Suppose that X_1^* and X_2^* are independent and
(i) $\bar{F}$ is strictly decreasing in $(0, \infty)$,
(ii) for each $x > 0$, $\frac{\bar{F}(x+t)}{\bar{F}(t)}$ is a monotone function of t,
then $X_1 \sim exponential(\lambda)$ for some $\lambda > 0$.

3 Some stochastic properties of unfailed components

In this section, we explore some stochastic and aging properties of unfailed components in the system.

(a) If F is NBU(NWU)[5], then $X^* \underset{st}{\leq} X \quad (X^* \underset{st}{\geq} X)$.

(b) If F is IFR (DFR), then $X^* \underset{hr}{\leq} X \quad (X^* \underset{hr}{\geq} X)$.

(c) If f is log-concave (log-convex), then $X^* \underset{lr}{\leq} X \quad (X^* \underset{lr}{\geq} X)$.

Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the signature vectors of two coherent systems with i.i.d. lifetimes distributed with the common continuous distribution function F. Denote by T_1 and T_2 the respective lifetimes of the systems. Let also X_1^* and X_2^* be the residual lifetimes of the unfailed components in two systems, respectively.

(a) If F is DFR and $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$, then $X_1^* \leq_{st} X_2^*$.

(b) If F is DFR and $\mathbf{s}_1 \leq_{hr} \mathbf{s}_2$, then $X_1^* \leq_{hr} X_2^*$.

(c) If f is log-convex and $\mathbf{s}_1 \leq_{lr} \mathbf{s}_2$, then $X_1^* \leq_{lr} X_2^*$.

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[5]New Better than Used(New Worse than Used)

On Testing Hazard Rates Order based of Type II Censoring

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Abstract

In this paper, a new test for testing hazard rate order between two independent populations based on right Type-II censoring samples is proposed and simulation study to evaluate the power of the proposed test are given.

Keywords and Phrases: Hazard rate ordering, Distribution-free test, Type-II censoring, Stochastic Ordering, Power, Placements, Precedence-type statistics.

AMS 2000 Subject Classification: primary 62G10; secondary 62N03, 62N05.

1 Introduction

The analysis of lifetime data is an area of statistics primarily concerned with making inference about life characteristics based on data from the occurrence of certain events. Often, they involve the occurrence of an undesirable event such as the failure of a component in a machine, breakage of a beam in a structure, and death of a patient. The large body of theory that has been developed for analyzing data of this kind is called reliability analysis (in the physical sciences and engineering) or survival analysis (in the biomedical sciences). In analyzing lifetime data, a problem that is often encountered is the comparison of reliability of two products. For example, a manufacturer of a product will naturally be interested in assessing the reliability of the product. It is common that the reliability of the product is evaluated by some characteristics of lifetimes such as mean, median, quantile, survival function and hazard rate function. Sometimes it is possible that the data are incomplete in the sense that some units had not failed by the time observation on them stopped. For such units, even though we do not know the exact value of the failure time T, we do have partial information about it. Since censoring is so commonly prevalent in lifetime data, statistical methods that have been developed for analyzing lifetime data do accommodate censoring in the model as well as in the subsequent analysis. In our discussion, we will focus primarily on *Type-II right censoring* as it is the most common form of censoring encountered in many practical problems. In reliability context, the most important inferential problems are comparisons of either reliability functions or hazard rate functions of two populations under various censorships, including Type-II censoring. Let X and Y be lifetime of two independent units with density functions f and g, distribution functions F and G, survival functions $F = 1 - F$ and $G = 1 - G$ and hazard rate functions $r_F = \frac{f}{F}$ and r_G respectively. In a particular life-testing experiment, suppose n_1 units with independent lifetimes $X_1, \cdots, X_{n_1}$ from X and n_2 units with independent lifetimes $Y_1, \cdots, Y_{n_2}$ from Y put on the test. Then, the experiment is terminated when the first r failures of X's are observed.
For the case of complete date, the problem of testing the hypotheses

$$H_0 : r_F(t) = r_G(t)$$

against

$$H_1 : r_F(t) \leq r_G(t) \text{ with strict inequality over a set of nonzero probability and } t \geq 0, \quad (1)$$

was first discussed in the literature by Chikkagoudar and Shuster (1974). Subsequently, many related developments have been made by Lee and Wolfe (1976), Kochar (1979, 1981), Joe and Proschan (1984) and Cheng (1985). In particular, Kochar (1981) proposed a test statistic based on the fact that H_1 is equivalent to

$$\eta(s,t) = \frac{\overline{F}(s)}{\overline{F}(t)} - \frac{\overline{G}(s)}{\overline{G}(t)} \geq 0 \quad for\ s \geq t \geq 0.$$

Taking $t = 0$, H_1 implies that $\overline{F}(s) \geq \overline{G}(s)$ for all $s \geq 0$, which is well known as *usual stochastic ordering* between X and Y. Kochar (1981) then defined the measure $\Gamma(F,G) = E\{\eta(X,Y)|X \geq Y\}$ $= 0.5 + \int_0^\infty F(x)\{0.5 + log\overline{G}(x)\}dG(x). It should be noted that under \mathrm{H}_0$, $\Gamma(F,G) = 0$ while under H_1, $\Gamma(F,G) > 0$.

In Section 2, we modify the following test statistic due to Kochar (1981), given by

$$S = \int_0^\infty F_{n_1}(x)\left[\frac{1}{2} + log\left\{1 - \frac{n_2}{n_2+1}G_{n_2}(x)\right\}\right]dG_{n_2}(x) \tag{2}$$

$$= \frac{1}{n_2}\sum_{j=1}^{n_2}\frac{R_{(j)} - j}{n_1}\left[\frac{1}{2} + log\left\{1 - \frac{j}{n_2+1}\right\}\right], \tag{3}$$

where F_{n_1} is the empirical version of F, G_{n_2} is empirical version of G and $R_{(j)}$ is the rank of $Y_{(j)}$, $j = 1,\ldots,n_2$, in the combined sample such that $Y_{(j)}$ is the jth order statistic of Y's. Note that H_0 in (1) rejected for large values of S. We establish and purpose three test statistics for the hypotheses testing problem in (1).

2 Main Results

Let the sample from F be Type-II censored on the right with the first r complete failures observed and the last $n_1 - r$ failures times censored. For the purpose of constructing meaningful test statistics in this case, we replace the distribution functions F and G in functional $\Gamma(F,G)$ with their suitable empirical versions F_{n_1} and G_{n_2} under censorship. We consider three different scenarios for the censored observations to be distributed in the combined sample. In all these scenarios, we start with the assumption that I random number of $Y's$ occur before $X_{(r)}$. Let $R_{(j)}$ denote the rank of $Y_{(j)}$ in the combined increasing arrangement of $X's$ and $Y's$, $R^*_{(j)} = R_{(j)} - j$, $a_j = \frac{1}{2} + \log\{1 - \frac{j}{n_2+1}\}$ and event $\{I = i\} \in \{0, 1, \cdots, n_2\}$.

Case 1: We assume that all remaining $(n_1 - r)$ X-failures occur immediately after $X_{(r)}$ and before $Y_{(I+1)}$. In this case, we obtain the test statistic

$$S_1(I) = \frac{1}{n_2 n_1}\sum_{j=1}^{I} R^*_{(j)} a_j + \frac{1}{n_2}\sum_{j=I+1}^{n_2} a_j.$$

Case 2: We suppose that all remaining $(n_1 - r)$ X-failures occur at the end after $Y_{(n_2)}$. In this case, we obtain the test statistic

$$S_2(I) = \frac{1}{n_1 n_2}\left[\sum_{j=1}^{I} R^*_{(j)} a_j + r\sum_{j=I+1}^{n_2} a_j\right].$$

Case 3: Since $S_1(I)$ and $S_2(I)$ are two extreme scenarios in terms of placements of X-observations, we propose the third test statistic by averaging $S_1(I)$ and $S_2(I)$, given by

$$S_E(I) = \frac{1}{n_2 n_1}\sum_{j=1}^{I} R^*_{(j)} a_j + \frac{1}{2n_1 n_2}[n_1 + r]\sum_{j=I+1}^{n_2} a_j.$$

It is evident that large values of $S_1(I)$, $S_2(I)$, and $S_E(I)$ lead to the rejection of H_0 in favor of H_1. The following theorem is used to find the exact null distribution of proposed test statistics. Under $H_0 : r_F(t) = r_G(t)$, for $1 \leq r_1 < \cdots < r_i \leq i + r - 1$, $\Pr\{I = i, R_{(1)} = r_1, R_{(2)} = r_2, \ldots, R_{(i)} = r_i | H_0\} =$

$\dfrac{\binom{n_1+n_2-r-i}{n_2-i}}{\binom{n_1+n_2}{n_2}}$. The p-value for observed values s_1, s_2, s_E of $S_1(I), S_2(I)$ and $S_E(I)$ can be readily computed. For example, we have

$$\begin{aligned}\Pr\left(S_1(I) \geq s_1|H_0\right) &= \frac{\binom{n_1+n_2-r}{n_2}}{\binom{n_1+n_2}{n_2}} I_{(-\infty, \frac{1}{n_2}\sum_{j=1}^{n_2} a_j)}(s_1) \\ &+ \sum_{i=1}^{n_2} \sum_{\substack{(r_1,\cdots,r_i),\\ 1\leq r_1<\cdots<r_i\leq i+r-1}} \frac{\binom{n_1+n_2-r-i}{n_2-i}}{\binom{n_1+n_2}{n_2}} I_{(s_1,\infty)}(s_1(i))\end{aligned}$$

where $I(.)$ is an indicator function.

In an analogous manner, we have the following expressions for p-values of $S_2(I)$ and $S_E(I)$.

$$\begin{aligned}\Pr(S_2(I) \geq s_2|H_0) &= \frac{\binom{n_1+n_2-r}{n_2}}{\binom{n_1+n_2}{n_2}} I_{(-\infty, \frac{r}{n_1 n_2}\sum_{j=1}^{n_2} a_j)}(s_2) \\ &+ \sum_{i=1}^{n_2} \sum_{\substack{(r_1,\cdots,r_i),\\ 1\leq r_1<\cdots<r_i\leq i+r-1}} \frac{\binom{n_1+n_2-r-i}{n_2-i}}{\binom{n_1+n_2}{n_2}} I_{(s_2,\infty)}(s_2(i))\end{aligned}$$

and

$$\begin{aligned}\Pr(S_E(I) \geq s_E|H_0) &= \frac{\binom{n_1+n_2-r}{n_2}}{\binom{n_1+n_2}{n_2}} I_{(-\infty, \frac{(n_1+r)}{2n_1 n_2}\sum_{j=1}^{n_2} a_j)}(s_E) \\ &+ \sum_{i=1}^{n_2} \sum_{\substack{(r_1,\cdots,r_i),\\ 1\leq r_1<\cdots<r_i\leq i+r-1}} \frac{\binom{n_1+n_2-r-i}{n_2-i}}{\binom{n_1+n_2}{n_2}} I_{(s_E,\infty)}(s_E(i))\end{aligned}$$

Next, we examine its power properties, through a Monte Carlo simulation study in which based on an extensive Monte Carlo simulation study, we evaluate the power of these new statistical tests and compare them with test statistics which discussed extensively in Balakrishnan and Ng (2006) (W_{min}, , W_{max} and W_E). Note that these test statistics compare distribution functions of two populations. Where in, based on 10000 replications of random samples from a particular distribution chosen from the alternative, we determined the *empirical power value* of the test for some specific alternatives. The power results for the tests $S_1(I)$, $S_2(I)$ and $S_E(I)$ and the tests W_{min}, W_{max} and W_E for the exponential, Gamma, Weibull and Makeham distributions for sample sizes 10 and 30 are presented in Tables 1 and 2, respectively. For this purpose, we took the null and alternative cases for these models as follows:

(**i**) Exponential with $E[X] = \theta$-$H_0 : X \overset{d}{=} Y \sim Exp(1)$ vs $H_1 : Y \sim Exp(1),\ X \sim Exp(\theta),\ \theta = 2, 4;$

(**ii**) *Gamma* with $E[X] = \gamma\theta$ and $Var(X) = \gamma\theta^2 - H_0 : X \overset{d}{=} Y \sim G(2,1)$ vs $H_1 : Y \sim G(2,1),\ X \sim G(2,\theta),\ \theta = 3, 6;$

(**iii**) Weibull with hazard function $r(x) = \frac{\gamma}{\theta}(\frac{x}{\theta})^{\gamma-1} - H_0 : X \overset{d}{=}\sim W(4,5)$ vs $H_1 : Y \sim W(4,5),\ X \sim$

$W(4,\theta),\ \theta = 6, 8$;
(iv) Makeham distribution with the survival function, $F_\theta(x) = \exp\left[-\{x + \theta(x + e^{-x} - 1)\}\right]$ where θ is a positive real number $-$ $H_0 : X \overset{d}{=} Y \sim Exp(1)$ vs $H_1 : X \sim Exp(1),\ Y \sim M(\theta),\ \theta = 3, 9$; Finally, on the base of power values, we concluded that the test based on S_E performs very well and is, therefore, the one that is recommended for testing the hazard rate ordering between two distributions. This makes the S_E-test proposed here to be quite important and helpful for testing the hazard rate ordering of two distributions.

Table1. **Power values of $S_1(I), S_2(I)$ and $S_E(I)$ and comparison with some precedence-type rank statistics for the case when $n_1 = n_2 = 10$.**

r	Test	$Exp(2)$	$Exp(4)$	$G(2,3)$	$G(2,6)$	$W(4,6)$	$W(4,8)$	$M(3)$	$M(9)$
5	$S_1(I)$	0.332	0.747	0.852	0.996	0.345	0.926	0.304	0.605
	$S_2(I)$	0.262	0.703	0.817	0.994	0.286	0.902	0.571	0.726
	$S_E(I)$	0.338	0.760	0.866	0.997	0.339	0.929	0.288	0.604
	W_{min}	0.275	0.674	0.853	0.994	0.281	0.878	0.043	0.172
	W_{max}	0.326	0.767	0.880	0.998	0.349	0.934	0.161	0.420
	W_E	0.325	0.760	0.886	0.997	0.351	0.929	0.086	0.270
6	$S_1(I)$	0.341	0.794	0.882	0.998	0.384	0.950	0.535	0.805
	$S_2(I)$	0.342	0.792	0.881	0.998	0.363	0.944	0.704	0.797
	$S_E(I)$	0.357	0.798	0.885	0.998	0.388	0.954	0.719	0.803
	W_{min}	0.327	0.749	0.872	0.996	0.348	0.914	0.054	0.202
	W_{max}	0.339	0.795	0.896	0.998	0.376	0.947	0.139	0.378
	W_E	0.329	0.766	0.886	0.998	0.352	0.937	0.087	0.268
7	$S_1(I)$	0.345	0.814	0.889	0.998	0.375	0.961	0.750	0.931
	$S_2(I)$	0.355	0.819	0.901	0.999	0.386	0.959	0.613	0.855
	$S_E(I)$	0.362	0.830	0.903	0.999	0.387	0.964	0.746	0.922
	W_{min}	0.322	0.761	0.882	0.997	0.361	0.926	0.062	0.217
	W_{max}	0.349	0.798	0.914	0.999	0.369	0.954	0.127	0.373
	W_E	0.350	0.782	0.899	0.998	0.371	0.941	0.087	0.270
8	$S_1(I)$	0.366	0.832	0.904	0.999	0.389	0.968	0.740	0.917
	$S_2(I)$	0.365	0.831	0.903	1.000	0.385	0.968	0.678	0.894
	$S_E(I)$	0.373	0.835	0.907	1.000	0.400	0.968	0.723	0.909
	W_{min}	0.341	0.781	0.891	0.998	0.368	0.935	0.077	0.235
	W_{max}	0.342	0.797	0.915	0.998	0.375	0.943	0.101	0.299
	W_E	0.354	0.799	0.909	0.998	0.388	0.948	0.088	0.269
9	$S_1(I)$	0.372	0.841	0.906	1.000	0.398	0.971	0.717	0.910
	$S_2(I)$	0.376	0.843	0.913	1.000	0.401	0.968	0.716	0.905
	$S_E(I)$	0.388	0.841	0.914	1.000	0.410	0.973	0.709	0.907
	W_{min}	0.333	0.781	0.911	0.998	0.357	0.934	0.080	0.265
	W_{max}	0.354	0.808	0.907	0.998	0.384	0.948	0.085	0.269
	W_E	0.359	0.809	0.911	0.999	0.393	0.945	0.086	0.272
10	$S_1(I)$	0.372	0.856	0.911	1.000	0.412	0.971	0.715	0.903
	$S_2(I)$	0.379	0.849	0.914	1.000	0.396	0.970	0.718	0.906
	$S_E(I)$	0.378	0.843	0.915	1.000	0.421	0.970	0.728	0.910
	W_{min}	0.341	0.778	0.908	0.998	0.357	0.935	0.084	0.270
	W_{max}	0.337	0.782	0.915	0.998	0.351	0.938	0.089	0.270
	W_E	0.335	0.778	0.907	0.998	0.355	0.937	0.097	0.294

Table2. Power values of $S_1(I), S_2(I)$ and $S_E(I)$ and comparison with some precedence-type rank statistics for the case when $n_1 = n_2 = 30$.

r	Test	$Exp(2)$	$Exp(4)$	$G(2,3)$	$G(2,6)$	$W(4,6)$	$W(4,8)$	$M(3)$	$M(9)$
15	$S_1(I)$	0.624	0.985	1.000	1.000	0.670	1.000	0.632	0.861
	$S_2(I)$	0.346	0.938	1.000	1.000	0.392	0.998	0.987	0.989
	$S_E(I)$	0.605	0.983	1.000	1.000	0.938	1.000	0.820	0.913
	W_{min}	0.564	0.971	0.999	1.000	0.600	0.999	0.014	0.105
	W_{max}	0.649	0.991	0.999	1.000	0.693	1.000	0.276	0.762
	W_E	0.645	0.989	0.999	1.000	0.680	1.000	0.107	0.535
20	$S_1(I)$	0.701	0.996	1.000	1.000	0.737	1.000	0.892	0.997
	$S_2(I)$	0.692	0.995	1.000	1.000	0.720	1.000	0.985	0.998
	$S_E(I)$	0.707	0.997	1.000	1.000	0.757	1.000	0.897	0.999
	W_{min}	0.666	0.992	0.999	1.000	0.671	1.000	0.050	0.380
	W_{max}	0.738	0.997	1.000	1.000	0.761	1.000	0.181	0.663
	W_E	0.691	0.995	1.000	1.000	0.732	1.000	0.107	0.531
23	$S_1(I)$	0.731	0.998	1.000	1.000	0.783	1.000	0.991	1.000
	$S_2(I)$	0.736	0.994	1.000	1.000	0.774	1.000	0.986	1.000
	$S_E(I)$	0.523	0.755	1.000	1.000	0.998	1.000	0.992	1.000
	W_{min}	0.700	0.996	1.000	1.000	0.744	1.000	0.079	0.463
	W_{max}	0.753	0.998	1.000	1.000	0.790	1.000	0.139	0.600
	W_E	0.723	0.998	1.000	1.000	0.753	1.000	0.100	0.517
25	$S_1(I)$	0.765	0.999	1.000	1.000	0.807	1.000	0.998	1.000
	$S_2(I)$	0.762	0.999	1.000	1.000	0.796	1.000	0.995	1.000
	$S_E(I)$	0.762	0.989	1.000	1.000	0.804	1.000	0.998	1.000
	W_{min}	0.713	0.998	1.000	1.000	0.751	1.000	0.080	0.474
	W_{max}	0.740	0.997	1.000	1.000	0.785	1.000	0.111	0.535
	W_E	0.746	0.997	1.000	1.000	0.775	1.000	0.097	0.509
27	$S_1(I)$	0.757	0.998	1.000	1.000	0.802	1.000	0.999	1.000
	$S_2(I)$	0.771	0.999	1.000	1.000	0.808	1.000	0.996	1.000
	$S_E(I)$	0.773	0.999	1.000	1.000	0.823	1.000	0.998	1.000
	W_{min}	0.723	0.997	1.000	1.000	0.754	1.000	0.092	0.505
	W_{max}	0.738	0.996	1.000	1.000	0.783	1.000	0.105	0.542
	W_E	0.726	0.996	1.000	1.000	0.761	1.000	0.100	0.524
30	$S_1(I)$	0.787	0.999	1.000	1.000	0.800	1.000	1.000	1.000
	$S_2(I)$	0.787	0.999	1.000	1.000	0.817	1.000	1.000	1.000
	$S_E(I)$	0.792	1.000	1.000	1.000	0.825	1.000	1.000	1.000
	W_{min}	0.745	0.997	1.000	1.000	0.768	1.000	0.095	0.510
	W_{max}	0.740	0.997	1.000	1.000	0.766	1.000	0.110	0.540
	W_E	0.739	0.997	1.000	1.000	0.769	1.000	0.098	0.514

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Bayesian estimators for Lomax distribution under generalized order statistics

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Abstract

The Bayesian parameter estimation, reliability and hazard functions for Lomax distribution are discussed based on generalized order statistics . The Bayesian estimators are in fact obtained based on conjugate prior for the shape parameter and discrete prior for the scale parameter of this distribution, with respect to both symmetric loss function (squared error) and asymmetric loss function (LINEX). Finally, the different Bayesian estimators are compared through simulation studies.

Keywords and phrases: Generalized order statistics, Lomax distribution, Bayesian estimator, Asymmetric loss function.

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1 Introduction

Generalized order statistics (gos) was introduced as a unified distribution theoretical set-up which contains a variety of approaches of ordered random variables with different interpretations [5]. They indeed play an important role in reliability theory and lifetime testing. Order statistics with non integral sample size, sequential order statistics, k-record values and progressively type II censoring, are particular cases of gos.

During the last decades some distributional properties of gos studied and minimum variance linear unbiased estimaes of the paramters of Lomax distribution obtained based on gos [3]. Furtheremore, the approaches of Bayesian and non-Bayesian esimation from Lomax distribution have been compared using record values [1]. Because of using the Lomax distribution for fitting business failure data, it has an important position in the field of lifetime testing, [4]. This family of distributions is useful for modeling and analyzing the lifetime data in medical and biological sciences, engineering, etc. Therefore, many statistical methods have been developed for this distribution.

Suppose that F be an absolutely continuous distribution function with density function f, and let $n \in N$, $k \geq 1$ and $\breve{m} = (m_1, \cdots, m_{n-1}) \in R^{n-1}$ be parameters such that $\zeta_r = k + n - r + \sum_{j=r}^{n-1} m_j \geq 1$ for all $r \in \{1, \cdots, n-1\}$. The random variables $X(r, n, \breve{m}, k)$, $r = 1, \cdots, n$, are said to be gos if their joint pdf is of the form

$$f^{X(1,n,\breve{m},k),\cdots,X(n,n,\breve{m},k)}(x_1, \cdots, x_n) = k \prod_{j=1}^{n-1} \zeta_j \left[\prod_{i=1}^{n-1} \overline{F}^{m_i}(x_i) f(x_i) \right] \overline{F}^{k-1}(x_n) f(x_n) \tag{1}$$

on the cone $F^{-1}(0) < x_1 \leq \cdots \leq x_n < F^{-1}(1)$, where $\overline{F}(x) = 1 - F(x)$, (see [5]).

For suitable choices of the parameters gos reduce to the well known ordered random variables, e.g. record values, progressively type II censoring. If $m_1 = \cdots = m_{n-1} = -1$ and $k = 1$, then $X(r, n, \breve{m}, k)$ reduces to the record values. If $m_i = R_i$ for $i = 1, \cdots, m-1,\ m_i = 0$, for $i = m, \cdots, n-1$ and $k = R_m + 1$, then

(1) gives the joint pdf of the progressively type II censoring samples.
The distribution function of Lomax distribution is given by

$$F(x;\alpha,\beta) = 1-(1+\frac{x}{\beta})^{-\alpha}, x>0, \quad \alpha,\beta>0, \tag{2}$$

where α and β are the shape and the scale parameters, respectively. The reliability function $R(t)$, and the hazard function $H(t)$ at time t for the Lomax distribution are respectively, given by

$$R(t) = (1+\frac{t}{\beta})^{-\alpha}, \quad H(t) = \frac{\alpha}{\beta}(1+\frac{t}{\beta})^{-1},\ t>0. \tag{3}$$

In life testing and reliability problems, the nature of losses are not always symmetric and hence the use of squared error loss function (SELF) is unacceptable in many situations. Therefore, an asymmetric linear exponential (LINEX) loss function introduces by [7]. Under the assumption that the minimal loss occurs at $\hat{\phi} = (\hat{\alpha},\hat{\beta})$, the LINEX loss function for $\phi = (\alpha,\beta)$ can be expressed as

$$L(\Delta) \propto e^{c\Delta} - c\Delta - 1; \quad c \neq 0, \tag{4}$$

where $\Delta = (\hat{\phi}-\phi)$ and $\hat{\phi}$ is an estimate of ϕ.
Up to now, Bayesian estimation method for Lomax distribution parameters has not been addressed under the gos. Therefore, in this work Bayesian estimators for parameters of Lomax distribution are derived based on gos, as well as for the survival time parameters. Specifically, in section 2 the Bayesian estimators are derived based on gos using the conjugate prior on the shape parameter and discretizing the scale parameter to a finite number of values. The estimators are obtained using both the SELF and LINEX loss function. Finally, we illustrate the performances of the considered estimators through simulated data.

2 Bayesian Estimators

Suppose that $X_{1,n,\breve{m},k}, X_{2,n,\breve{m},k}, \cdots, X_{n,n,\breve{m},k}, k \geq 1$, are n gos based on the density function from Lomax distribution. According to (1) and (2), the likelihood function can be obtained

$$L(\alpha,\beta;\mathbf{x}) = k(\frac{\alpha}{\beta})^n \left[\prod_{j=1}^{n-1}\gamma_j\right] e^{-\alpha u - v}, \tag{5}$$

where

$$u = \sum_{i=1}^{n-1}\ln(1+\frac{x_i}{\beta})^{m_i+1} + k\ln(1+\frac{x_n}{\beta}), \quad v = \sum_{i=1}^{n}\ln(1+\frac{x_i}{\beta}). \tag{6}$$

2.1 Known scale parameter

In case where β is known, we assume a gamma (a, b) conjugate prior for α as

$$\pi(\alpha|a,b) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}e^{-b\alpha},\ a,b>0. \tag{7}$$

Combining the likelihood function (5) and the latter prior, we can obtain the posterior density of α given the data

$$\pi(\alpha|\mathbf{x},\beta) = \frac{(b+u)^{n+a}}{\Gamma(n+a)}\alpha^{n+a-1}e^{-\alpha(b+u)}, \tag{8}$$

where u is defined as (6).
Under the SELF, Bayesian estimator of α is given by

$$\hat{\alpha}_{BS} = E(\alpha|\mathbf{X}) = \frac{n+a}{b+u}. \tag{9}$$

Similarly, the Bayesian estimator for the reliability function $R(t)$ and the hazard function $H(t)$ with fixed $t > 0$ is given by

$$\hat{R}_{BS}(t) = \left\{1 + \frac{\ln(1+\frac{t}{\beta})}{b+u}\right\}^{-(n+a)}, \quad \hat{H}_{BS}(t) = \frac{n+a}{(b+u)(t+\beta)}. \tag{10}$$

Alternatively, under the LINEX loss function the Bayesian estimator of α, is given by

$$\hat{\alpha}_{BL} = -\frac{1}{c}\ln\left[\int_0^\infty e^{-c\alpha}\pi(\alpha|\mathbf{x},\beta)d\alpha\right] = \frac{n+a}{c}\ln\left[1+\frac{c}{b+u}\right]. \tag{11}$$

Moreover, the Bayesian estimators for $R(t)$ and $H(t)$ are given by

$$\hat{R}_{BL}(t) = -\frac{1}{c}\ln\left[\sum_{i=0}^{\infty}\frac{(-c)^i}{i!}\left\{1+\frac{i\ln(1+\frac{t}{\beta})}{b+u}\right\}^{-(n+a)}\right], \tag{12}$$

$$\hat{H}_{BL}(t) = \frac{n+a}{c}\ln\left\{1+\frac{c}{(b+u)(t+\beta)}\right\}. \tag{13}$$

2.2 Unknown scale and shape parameters β and α

Determining a general joint prior for α and β in case where both are unknown initially requires specifying a discrete or continuous pdf on one of the α and β. In the literature of Bayesian analysis, Soland's method is used [6]. This method involves a family of joint prior pdfs that places a continuous pdf on α and a discrete distribution on β.

We assume that the scale parameter β is restricted to a finite number of values $\beta_1, \beta_2, \cdots, \beta_k$ with respective prior probabilities $\psi_1, \psi_2, \cdots, \psi_k$ such that $0 \le \psi_j \le 1$, and $\sum_{j=1}^k \psi_j = 1$. [i.e. $\Pr(\beta = \beta_j) = \psi_j$]. Further, suppose that conditional upon $\beta = \beta_j$, α has a natural conjugate prior with distribution gamma(a_j, b_j) with density

$$\pi(\alpha|\beta=\beta_j) = \frac{b_j^{a_j}}{\Gamma(a_j)}\alpha^{a_j-1}e^{-b_j\alpha}, \ a_j, b_j > 0. \tag{14}$$

Then given the set of n gos values $\mathbf{x}$, the conditional posterior pdf of α is

$$\pi(\alpha|\beta=\beta_j,\mathbf{x}) = \frac{(b_j+u_j)^{n+a_j}}{\Gamma(n+a_j)}\alpha^{n+a_j-1}e^{-\alpha(b_j+u_j)}, \ \alpha, a_j, b_j > 0, \tag{15}$$

with $u_j = \sum_{i=1}^{n-1}\ln(1+\frac{x_i}{\beta_j})^{m_i+1} + k\ln(1+\frac{x_n}{\beta_j})$.

The marginal posterior mass function of β_j is obtained as

$$P_j = Pr(\beta=\beta_j|\mathbf{x}) = A\frac{\psi_j e^{-v_j} b_j^{a_j}\Gamma(n+a_j)}{\beta_j^n(b_j+u_j)^{n+a_j}\Gamma(a_j)}, \tag{16}$$

where $v_j = \sum_{i=1}^n \ln(1+\frac{x_i}{\beta_j})$ and A is the normalized constant given by

$$A^{-1} = \sum_{j=1}^k \frac{\psi_j e^{-v_j} b_j^{a_j}\Gamma(n+a_j)}{\beta_j^n(b_j+u_j)^{n+a_j}\Gamma(a_j)}. \tag{17}$$

The Bayesian estimators of α and β under the SELF can then be obtained using the posterior pdfs (15) and (16) as the form

$$\hat{\alpha}_{BS} = \sum_{j=1}^k \frac{P_j(n+a_j)}{(b_j+u_j)}, \quad \hat{\beta}_{BS} = \sum_{j=1}^k P_j\beta_j. \tag{18}$$

Similarly, the Bayesian estimators of survival functions $R(t)$ and $H(t)$ with fixed $t > 0$ are given respectively by

$$\hat{R}_{BS}(t) = \sum_{j=1}^{k} P_j \left(1 + \frac{\ln(1 + \frac{t}{\beta_j})}{b_j + u_j}\right)^{-(n+a_j)}, \quad \hat{H}_{BS}(t) = \sum_{j=1}^{k} P_j \frac{(n + a_j)}{(b_j + u_j)(t + \beta_j)}. \tag{19}$$

Alternatively, under the LINEX loss function, the Bayesian estimator for the α and β can be derived

$$\hat{\alpha}_{BL} = -\frac{1}{c} \ln \left[\sum_{j=1}^{k} P_j \left(1 + \frac{c}{b_j + u_j}\right)^{-(n+a_j)}\right], \quad \hat{\beta}_{BL} = -\frac{1}{c} \ln \left[\sum_{j=1}^{k} P_j e^{-c\beta_j}\right]. \tag{20}$$

Similarly, the Bayesian estimator for the reliability function $R(t)$ and $H(t)$ with fixed $t > 0$ are respectively given by

$$\hat{R}_{BL}(t) = -\frac{1}{c} \ln \left[\sum_{j=1}^{k} \sum_{i=0}^{\infty} P_j \frac{(-c)^i}{i!} \left\{1 + \frac{i \ln(1 + \frac{t}{\beta_j})}{b_j + u_j}\right\}^{-(n+a_j)}\right], \tag{21}$$

$$\hat{H}_{BL}(t) = -\frac{1}{c} \ln \left[\sum_{j=1}^{k} P_j \left\{1 + \frac{c}{(b_j + u_j)(t + \beta_j)}\right\}^{-(n+a_j)}\right]. \tag{22}$$

Remark: By choosing $m_i = -1, i = 1, \cdots, n - 1$ and $k = 1$, the above results in this section, we obtain Bayesian estimators corresponds to the nth upper record from Lomax distribution which coincides with that obtained by [1].

3 Simulation study and comparisons

In order to compare performances of the Bayesian estimators, a simulation study are used based on the algorithm presented in [2] according to the following steps in the case where β is known.

1. For given values (a=3, b=2) generate $\alpha = 2.32$ from the prior pdf (7).
2. Using the value $\alpha = 2.32$ from step 1, with known $\beta = 2$ and by choosing the parameters $m_i = R_i$ for $i = 1, \cdots, m-1,\ m_i = 0$, for $i = m, \cdots, n-1$ and $k = R_m + 1$ in the mentioned algorithm, and with predetermined censoring scheme, we generate a progressively type II censored sample from the Lomax pdf. Table 1 shows the generated samples.

Table1. Progressively type II censored sample when β is known

j	1	2	3	4	5	6	7	8	9	10
$X_{j:m:n}$	0.028	0.069	0.196	0.22	0.444	0.49	0.631	1.082	1.542	1.731
R_j	1	0	1	2	0	0	3	0	1	2

Table2. Estimators of α, $R(t)$ and $H(t)$ when β is known

Samples	Parameters	$(.)_{BS}$	$(.)_{BL}$		
Recorded values			$c=-2$	$c=-1$	$c=1$
	α	2.089	2.521	2.278	1.937
	$R(t)$	0.253	0.263	0.258	0.248
	$H(t)$	0.522	0.544	0.533	0.512
			$c=-1$	$c=1$	$c=5$
Progressively type II censored sample	α	3.056	3.279	2.869	2.343
	$R(t)$	0.132	0.133	0.130	0.124
	$H(t)$	0.764	0.777	0.752	0.707

3. Using the value $\alpha = 2.32$ from step 1, with known $\beta = 2$ and by choosing the parameters $m_1 = \cdots = m_{n-1} = -1$ and $k = 1$ in the mentioned algorithm, we generate n, (n=10) upper record value from the Lomax pdf as
   0.3175, 1.0717, 4.67, 9.448, 18.708, 29.955, 39.286, 70.98, 82.687, 134.157

4. The Bayesian estimators of α, $R(t)$ and $H(t)$ (for $t = 2$) under SELF and LINEX loss functions are computed by using the results in section 2. The results are displayed in Table 2.

4 Conclusion

Based on gos, this paper proposed Bayesian approach to estimate two unknown parameters in the Lomax distribution, as well as the reliability and hazard functions. We developed the results of [1] in which Bayesian estimators from the Lomax distribution has been derived using record values. The estimators obtained using both the symmetric and asymmetric loss functions. Comparisons are made between different estimators based on simulation study. In case of known β, Table 2 shows that the Bayesian estimators under the LINEX loss function are doing better than the ones under the SELF for both record values and progressively type II censoring. The asymmetric Bayesian estimators are more relatively sensitive to the values of the shape parameter c of the LINEX loss function.

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On the bathtub shaped hazard rate functions

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Abstract

In this paper, some results that are concerned with the class of distributions with bathtub shaped hazard rate functions has been reviewed. It is known that like some other reliability measures, the mean residual life function has upside down bathtub shape when the hazard rate function has bathtub form. One interesting problem arising for lifetime models with bathtub shape is burn-in, and is extensively studied in the reliability literature. Here, some results about the burn-in topic have been presented. It has been noticed that the hazard rate function of the order statistics may be increasing, bathtub shaped, etc. Then, two criteria for burn-in have been considered, and the optimal burn-in time of the order statistics have been compared with the corresponding optimal burn-in time of the baseline distribution in terms of their locations.

Key words and phrases: Bathtub shaped hazard rate, Reliability measure, Burn-in, Order statistics.

AMS Subject Classification 2000: primary 68M15; secondary 62G30.

1 Introduction

One of the well-known reliability measures that characterizes the lifetime distribution is the hazard rate function, $h(x)$. It is interpreted as the instantaneous risk of failure at time $x > 0$ given survival up to x. In other words, $h(x)dx$ is the probability of failure at $(x, x + dx]$ given that it has survived time x. The hazard rate function is closely related to other reliability measures like the reliability function, the mean residual lifetime (MRL), the α-percentile residual lifetime (α-PRL), etc.

In many practical situations, there are objects (manufactured objects or born creatures) exposed to an excessive risk in an early period of their life, but improve with time until a stable condition called useful period. When the useful period elapses, the object enter the wear out period in which the failure is caused by erosion or fatigue, and risk of failure grows with time during it. In attention to model such aging phenomena, it is preferred to apply a model that accommodates bathtub shaped hazard rate functions rather than a model with just monotone hazard rate functions. Many authors introduced examples of lifetime distributions capable of modeling bathtub shaped hazard rate functions in the literature. Among them we can point out Glaser (1980), Mudholkar and Sirvastava (1993), Navarro and Hernandez (2004), Xie et al. (2002), Wang (2000) and many others. Also, Lai and Xie (2006) and Nadarajah (2009) provided lists of such models.

The class of bathtub shaped hazard rate functions has been received considerable attention in both practical and theoretical aspects of reliability by many authors. For two good reviews about this class of distributions, one can refer to Rajarshi and Rajarshi (1988) and Lai et al. (2001).

Since objects with BT hazard rate are exposed to high failure rate in early stage of life, they can be tested to screen out damaged or unreliable objects before putting them to field operation or customers, i.e., the objects pass a burn-in period to reach more stable conditions. The burn-in topic has been received many attention in the reliability literature. Many authors have been studied the optimal burn-in time under different criteria and different assumption, e.g., we can refer to Mi (1994, 1995, 1997), Block et al. (1999), Block et al. (2002), Cha (2005, 2006), Sheu and Chien (2004, 2005), Bebbington et al. (2007), Kim and Kuo (2009) and Cha and Finkelstein (2010).

Many authors studied reliability properties of the order statistics and systems specially for the class of distributions with monotonic hazard rate functions. Among them we can refer to Navarro and Hernandez (2008), Eryilmaz et al. (2009), Asadi and Bairamov (2005), Asadi and Goliforushani (2008), Kundu et

al. (2009), Zhang and Yang (2010), Nanda et al. (2010), Khaledi et al. (2011) and many others. Shen et al. (2010) obtained relationships between the MRL change points of the series and parallel systems and the change points of the MRL function of components. Here, we presents results that are more general than some of their findings, but the assumptions in these two approaches have trifle differences.
In this paper, we review some results concerned with the class of BT hazard rate distributions. It has been shown that the MRL function is upside down bathtub shaped (UBT) when the hazard rate function exhibits bathtub shape. Then, we briefly review some results about burn-in. We consider that for models with BT hazard rate functions the hazard rate function of order statistics may be increasing, BT, etc. Then, two burn-in criteria have been considered, and the optimal burn-in time of a distribution has been compared with the optimal burn-in time of its corresponding order statistics in terms of their locations.

2 MRL function and BT hazard rate function

Let lifetime of an object is represented by the random variable T following the distribution function $F(x)$ and the reliability function $\bar{F}(x)$, $x \in (0, u)$, $0 < u \leq \infty$. The conditional remaining lifetime of the object given survival up to time x is denoted by $T_x = (T - x | T \geq x)$. If F be absolutely continuous and $f(x)$ be its density function, then the hazard rate function and the MRL function are respectively defined by

$$h(x) = \frac{f(x)}{\bar{F}(x)},$$

and

$$m(x) = E(T_x) = \frac{\int_0^x \bar{F}(t)dt}{\bar{F}(x)},$$

that are related to each other by

$$h(x) = \frac{m'(x) + 1}{m(x)},$$

which obviously demonstrates that $m'(x) \geq -1$. Mi (1995) and Gupta and Akman (1995) studied the MRL function of BT hazard rate distributions, and obtained similar results. We recall the following definition from Mi (1995).

Definition 1. *A real valued function $g(x)$, $x \in [0, \infty)$ has a bathtub (upside down bathtub) shape if there exist $0 \leq x_1 \leq x_2 \leq \infty$ such that $g(x)$ strictly decreases (increases) for $x \in [0, x_1]$, is constant on $[x_1, x_2]$, and strictly increases for $x \geq x_1$.*

x_1 and x_2 are usually called change points of g. Clearly, based on this definition the class of BT functions consists of class of increasing and decreasing functions. Mi (1995) proved the following theorem that determines shape of the MRL function when the hazard rate function is BT. Let F have a differentiable hazard rate function h that is BT with change points x_1 and x_2.
(i) If $0 < x_1 \leq x_2 < \infty$, then $m(x)$ is BT with a unique change point $x_* \in [0, x_1]$.
(ii) If $0 = x_1 \leq x_2 < \infty$, then $m(x)$ strictly decreases, i.e., there is a unique change point at $x = 0$.
(iii) If $x_1 = x_2 = \infty$, then $m(x)$ strictly increases.
(iv) If $0 < x_1 < x_2 = \infty$, then $m(x)$ strictly increases on $[0, x_1]$, and is constant on $[x_1, \infty)$. Block et al. (2002) considered a more general problem, and studied behavior of the average hazard rate function, the MRL function, the average mean residual life function, the harmonic average mean residual life function, the variance residual life function and the dynamic entropy function.

3 Burn-in

Mi (1994) considered an BT hazard rate object that must pass a given mission time $\tau > 0$, and defined the burn-in time b that maximizes the reliability for this mission time as the optimal burn-in time, i.e. for a given mission time $\tau > 0$, b is the optimal burn-in time if

$$\frac{\bar{F}(b+\tau)}{\bar{F}(b)} = \max_{x \geq 0} \frac{\bar{F}(x+\tau)}{\bar{F}(x)}. \quad (1)$$

But, the optimal burn-in time in this approach is not unique always, so he defined the set of all optimal burn-in times as

$$B = \{b : \frac{\bar{F}(b+\tau)}{\bar{F}(b)} = \max_{x\geq 0} \frac{\bar{F}(x+\tau)}{\bar{F}(x)}\}. \tag{2}$$

(Mi 1994a). Assume that $h(x)$ is an BT hazard rate function with the change points x_1 and x_2, and $\tau > 0$ be a given mission time.
(i) If $\tau \leq x_2 - x_1$, then $B = [x_1, x_2 - \tau]$.
(ii) If $\tau > x_2 - x_1$, then $B = b$, and $b \in [0, x_1]$. Let $b^* = \inf B$ that is reasonable in many cases, then $b^* \leq x_1$. One of the important measures for quality of an object is the length of time that it works properly. With this idea, Mi (1995) considered $b > 0$ that maximizes the MRL function as the optimal burn-in time, i.e., b is the optimal burn-in time iff

$$m(b) = \max_{x\geq 0} m(x). \tag{3}$$

Then, he utilized Theorem 4.1 to derive the following results for an BT hazard rate lifetime model.
1. When $x_1 = 0$, burn-in is not needed, i.e., $b = 0$.
2. if $x_2 = \infty$ and $x_1 > 0$, then $b = x_1$.
3. if $x_1 = x_2 = \infty$ that is F has strictly decreasing hazard rate function, then the cost determines the burn-in time.
4. In the case that $0 < x_1 \leq x_2 < \infty$, the burn-in time b equals the the unique point that maximizes the MRL function.

4 Order statistics of BT hazard rate distributions

Suppose that X_i, $i = 1, 2, ..., n$ are n lifetimes and $X_{i:n}$ is the i^{th} order statistics. Usually, the lifetime of a $n - k + 1$-out-of-n system composed of n independent and identically distributed (iid) elements X is represented by $X_{k:n}$. The hazard rate function of $X_{k:n}$ is given by

$$h_{k:n}(x) = k\binom{n}{k} g_{k:n}(t(x))h(x), \tag{4}$$

where $t(x) = \frac{F(x)}{\bar{F}(x)}$ and $g_{k:n}(x) = \frac{x^{k-1}}{\sum_{i=0}^{k-1}\binom{n}{i}x^i}$.

Definition 2. *(Shafaei et al. 2011a). a lifetime distribution function $F(x)$, $0 \leq x < u$, $0 < u \leq \infty$ is said to be BT hazard rate with the change point x_0 when h is strictly decreasing at $(0, x_0)$, constant in (x_0, x_1) for a $x_1 \in (x_0, u)$ and strictly increasing in (x_1, u).*

Mitra and Basu (1996) studied some properties of the class of BT hazard rate lifetime distributions. They presented some examples to illustrate that this class is not closed under convolution, coherent systems or mixtures. Also, Shafaei et al. (2011a) studied the hazard rate of the order statistics corresponding to an BT hazard rate function, and illustrated that it is eventually increasing, and may be increasing, BT, etc. The following simple result indicates that when the hazard rate function of X attains finite value at zero, the order statistics do not exhibit an BT hazard rate function. (i) If $h_{k:n}(.)$, $k \geq 2$ is BT, then $h(0) = \infty$.
(ii) If $h_{k+1:n}(.)$ or $h_{k+1:n+1}(.)$, $k \geq 1$ are BT, then $h_{k:n}(0) = \infty$. Furthermore, they proved the following results concerning the shape of the hazard rate function of order statistics. Suppose that $F_X(.)$ is BT_{x_0}. If for every $x \in (0, x_0)$, we have

$$h'(x) = \frac{dh(x)}{dx} \geq -\frac{h^2(x_0)\xi_{k:n}(t(x_0))}{F(x_0)}, \tag{5}$$

where $\xi_{k:n}(x) = \frac{\sum_{i=0}^{k-1}(k-1-i)\binom{n}{i}x^i}{\sum_{i=0}^{k-1}\binom{n}{i}x^i}$, and $t(x) = \frac{F(x)}{\bar{F}(x)}$, then $F_{k:n}(.)$ is IHR. Let $F_X(.)$ be BT_{x_0}. If there exists x_* such that $x_* \leq x_0$, and for every $x \in (x_*, x_0)$ we have

$$h'(x) \geq -\frac{h^2(x_0)\xi_{k:n}(x_0)}{F(x_0)},$$

and for every $x \in (0, x_*)$ we have

$$h'(x) \leq -(k-1)\frac{h^2(x)}{F(x)},$$

then $F_{k:n}(.)$ is an BT_{x_*} distribution function.

Example 6.2: Murthy and Jiang (1997) studied a sectional Weibull models with the hazard rate function given by

$$h(x) = \begin{cases} ab(ax)^{b-1}, & 0 \leq x \leq x_0, \\ cd(cx)^{d-1}, & x_0 \leq x, \end{cases}$$

where $x_0 = (\frac{b}{d}\frac{a^b}{c^d})^{\frac{1}{d-b}}$. When $b < 1$ and $d > 1$, the hazard rate exhibits a bathtub shape with the change point x_0. As a special case, let $a = c$. Shafaei et al. (2011a) showed that when $b \leq 0.3873$, $h_{2:n}(.)$ has bathtub shaped with the change point x_0.

4.1 Optimal burn-in time of order statistics

Shafaei et al. (2011b) compared the optimal burn-in time of a k-out-of-n system with the one related to the components based on the conditional reliability for a given mission time $\tau > 0$ criterion defined by Mi (1994) and the MRL criterion studied in Mi (1995). Firstly, consider the following lemma. (Shafaei et al. 2011b). Let F_1 and F_2 be two BT hazard rate distribution functions with hazard rate functions $h_1(x)$ and $h_2(x) = h_1(x)\psi(x)$, where $\psi(.)$ is an increasing function. Let τ be a fixed positive constant, and B_1 and B_2 are the sets of all optimal burn-in points, defined in (3.2), corresponding to h_1 and h_2, respectively. Then we have

$$\inf B_2 \leq \inf B_1.$$

Let $F(.)$ and $F_{k:n}(.)$ are BT hazard rate distribution functions, and B and $B_{k:n}$ are the sets of all optimal burn-in times , given in (3.2), corresponding to $F(.)$ and $F_{k:n}(.)$, respectively, then

$$\inf B_{k:n} \leq \inf B.$$

Suppose that $F_{k:n}(.)$, $F_{k+1:n}(.)$, $F_{k:n-1}(.)$, and $F_{k+1:n+1}(.)$ are BT hazard rate distribution functions, and respectively, $B_{k:n}$, $B_{k+1:n}$, $B_{k:n-1}$, and $B_{k+1:n+1}$ are their corresponding sets of all optimal burn-in times, determined in (3.2), then we have:
(i) $\inf B_{k+1:n} \leq \inf B_{k:n}$,
(ii) $\inf B_{k:n-1} \leq \inf B_{k:n}$,
(iii) $\inf B_{k+1:n+1} \leq \inf B_{k:n}$. In the following results the MRL function has been considered as the burn-in criterion. Let $F_1(x)$ and $F_2(x)$, $x \in (0, \infty)$ be two BT hazard rate distribution functions with the hazard rate functions $h_1(x)$ and $h_2(x) = \psi(x)h_1(x)$, and x_0 and x_* are the MRL change points of $F_1(.)$ and $F_2(.)$, given in (3.3), respectively.
(i) If $\psi(x)$ is increasing, and $0 \leq \psi(x) \leq 1$, then $x_* \leq x_0$.
(ii) If $\psi(x)$ is decreasing, and $\psi(x) \geq 1$, then $x_* \geq x_0$.
The following theorem indicate that the burn-in time of a parallel system precedes the burn-in time of its components. Consider F and $F_{k:n}$ be two BT hazard rate distributions, and x_0 and $x_{k:n}$ be their MRL burn-in times respectively. If $k = n$, then we have

$$x_{k:n} \leq x_0.$$

Let $F_{k:n}$, $F_{k+1:n}$, $F_{k:n-1}$, and $F_{k+1:n+1}$ be BT distributions, and take $x_{k:n}$, $x_{k+1:n}$, $x_{k:n-1}$, and $x_{k+1:n+1}$ as their MRL burn-in times respectively. Then, we have:
(i) $x_{k+1:n} \leq x_{k:n}$,
(ii) $x_{k:n-1} \leq x_{k:n}$,
(iii) $x_{k+1:n+1} \leq x_{k:n}$.

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A new three-parameter lifetime distribution

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Abstract

In this paper, a new three-parameter lifetime distribution is introduced by combining an extended Exponential distribution with a Geometric distribution. This flexible distribution has increasing, decreasing and upside-down bathtub failure rate shapes. Various properties of the proposed distribution are discussed. The estimation of the parameter attained by the EM algorithm are obtained.

Keywords and Phrases: Extended Exponential distribution, EM algorithm, Lifetime distribution, Maximum likelihood estimation

AMS Subject Classification 2000: primary 60K10.

1 Introduction

The literature of statistics has numerous distributions for modeling lifetime data. But many, if not most of these distributions, lack motivation from a lifetime context. For example, there is no apparent physical motivation for the gamma distribution. It only has a more general mathematical form than the exponential distribution with one additional parameter, so it has more appropriate properties and provides better fit. The same arguments apply to Weibull and many other distributions.

The aim of this paper is to introduce a new three parameter lifetime distribution with strong physical motivation.

The latter distribution can be obtained by considering the lifetime of a parallel system with Z identical components, where failure occurs when all omponents cease to function, i. e., the lifetime of the system is $X = \max\{Y_1, Y_2, \dots Y_z\}$. In this paper, we assume that, the lifetimes of the components are exponentiated exponential independent random variables and the distribution of their number is Geometric.

Suppose a company has Z parallel systems functioning independently at a given time, where Z is a geometric random variable with the probability mass function

$$P(z;p) = p^{z-1}(1-p) \tag{1}$$

for $z = 1, 2, \dots$. In addition, suppose that each system is made of α parallel units, so the system will fail if all of the units fail. Assume, the failure times of the units for the ith system, $Z_{i1}, Z_{i2}, \dots, Z_{i\alpha}$, are independent and identical exponential random variables with the scale parameter β. Let Y_i denotes the failure time of ith system with density

$$f(y;\alpha,\beta) = \alpha\beta\exp(-\beta y)(1-\exp(-\beta y))^{\alpha-1}.$$

Let X denotes the time to failure of the last out of the Z functioning systems. We can write $X = \max\{Y_1, Y_2, \dots, Y_z\}$. Then the probability density function of X, say $f(x)$, can be derived as: the conditional probability density function of X given Z is $f(x|z) = \alpha\beta z\exp(-\beta x)(1-\exp(-\beta x))^{\alpha z-1}$ and $f(x,z) = \alpha\beta z p^{z-1}(1-p)\exp(-\beta x)(1-\exp(-\beta x))^{\alpha z-1}$.

So, the marginal density function of X is

$$f(x) \quad = \quad \alpha\beta(1-p)\exp(-\beta x)\frac{\Big(1-\exp(-\beta x)\Big)^{\alpha-1}}{\Big(1-p(1-\exp(-\beta x))^{\alpha}\Big)^2} \tag{2}$$

Figure 1 shows the graph of density function of (2) for $\beta = 1$ and selected values of α and p.

The cumulative distribution function corresponding to (2) is given by

$$F(x) \quad = \quad \frac{(1-p)(1-\exp(-\beta x))^{\alpha}}{1-p(1-\exp(-\beta x))^{\alpha}} \tag{3}$$

for $x > 0$, $0 < p < 1$, $\alpha > 0$ and $\beta > 0$. We shall refer to the distribution given by (3) as the *max exponentiated exponential geometric* (max EEG) distribution.

The max EEG distribution can be motivated in many other ways, too. Here, we present three other ways to motivate (3).

Suppose a company has Z parallel systems functioning independently at a given time, where Z is a geometric random variable with the probability mass function given by (1). Suppose that the lifetime of each system is an exponentiated exponential random variable given by the cumulative distribution function $(1-\exp(-\beta x))^{\alpha}$ for $x > 0$, $\alpha > 0$ and $\beta > 0$. (The exponentiated exponential distribution due to Gupta and Kundu (1999, 2001) has received widespread attention.) Then the time to failure of the last out of the Z functioning systems will have the cumulative distribution function given by (3).

For a third motivation, using the Taylor series expansion

$$(1-px)^{-a} = \sum_{k=0}^{\infty} \binom{-a}{k} (px)^k$$

(3) can be written as

$$F(x) = (1-p)\sum_{k=0}^{\infty}(-p)^k(1-exp(-\beta x)^{\alpha k+\alpha} \tag{4}$$

We can see that the max EEG is a mixture of exponentiated exponential distributions.

2 Hazard Rate Function

The hazard rate function of the max EEG distribution is given by

$$h(x) \quad = \quad \frac{\alpha\beta(1-p)\exp(-\beta x)(1-\exp(-\beta x))^{\alpha-1}}{\Big(1-(1-\exp(-\beta x))^{\alpha}\Big)\Big(1-p(1-\exp(-\beta x))^{\alpha}\Big)} \tag{5}$$

for $x > 0$, $\alpha > 0$, $0 < p < 1$ and $\beta > 0$.

Figure 2 shows possible shapes of (5) for $\beta = 1$ and selected values of p and α. In the following, we examine the behaviors of hazard rate function.

$$\frac{\partial h(x)}{\partial x} \quad = \quad -\frac{\beta\Big(1-\exp(-\beta x)\Big)^{\alpha}\Big(-\alpha+\exp(\beta x)\Big)}{\Big(-1+\exp(\beta x)\Big)^{2}\Big(-1+(1-\exp(-\beta x))^{\alpha}\Big)\Big(-1+p(1-\exp(-\beta x))^{\alpha}\Big)} \tag{6}$$

So, the hazard rate function is decreasing for $\frac{1}{\alpha} < \exp(-\beta x)$.

3 Median

The median of a random variable with the max EEG distribution can be easily obtained by using (3). this is obvious

$$Median(X) = \frac{-1}{\beta}\ln\Big(1-(\frac{1}{2-p})^{\frac{1}{\alpha}}\Big).$$

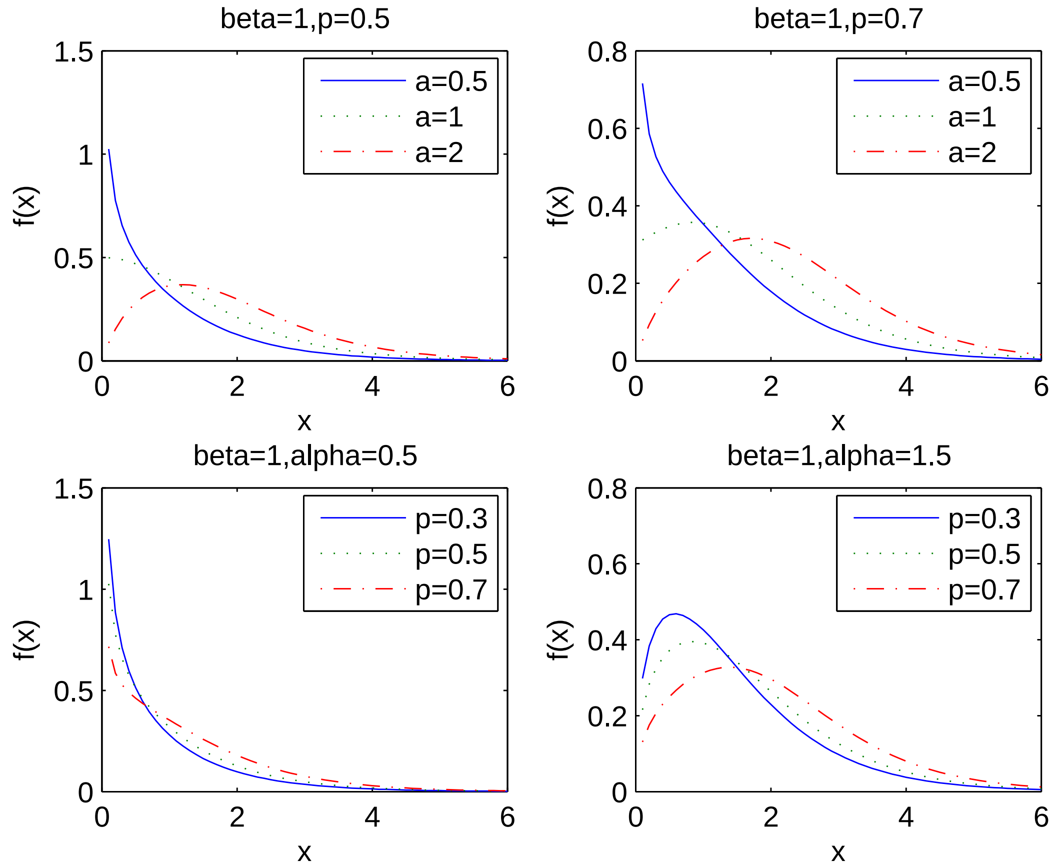

Figure 1: Probability density function of the max EEG distribution.

4 Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample from the max EEG distribution. Let $X_{i:n}$ denote the ith order statistic. The probability density function of $X_{i:n}$ is

$$f_{i:n}(x) \quad = \quad \frac{z!}{(i-1)!(z-i)!}\alpha\beta(1-p)^i\exp(-\beta x)\frac{\Big(1-\exp(-\beta x)\Big)^{\alpha i-1}\Big(1-(1-\exp(-\beta x))^{\alpha}\Big)^{z-i}}{\Big(1-p(1-\exp(-\beta x))^{\alpha}\Big)^{z+1}}$$

5 Moments

The rth moment of a random variable can be easily obtained by using (4). We have

$$E(X^r) = (1-p)\sum_{k=0}^{\infty}(-p)^k E(Y^r_{\beta,\alpha k+\alpha}). \tag{7}$$

where $Y^r_{\beta,\alpha k+\alpha}$ is an exponentiated exponential random variable with parameters α and β. Gupta and Kundu (1999, 2001) showed that the rth moment of $Y_{\beta,\alpha}$ can be expressed as

$$E(Y^r_{\beta,\alpha k+\alpha}) = \frac{\alpha r!}{\beta^r}\sum_{i=0}^{\infty}(-1)^i\binom{\alpha-1}{i}\frac{1}{(i+1)^{r+1}}.$$

So, we obtain

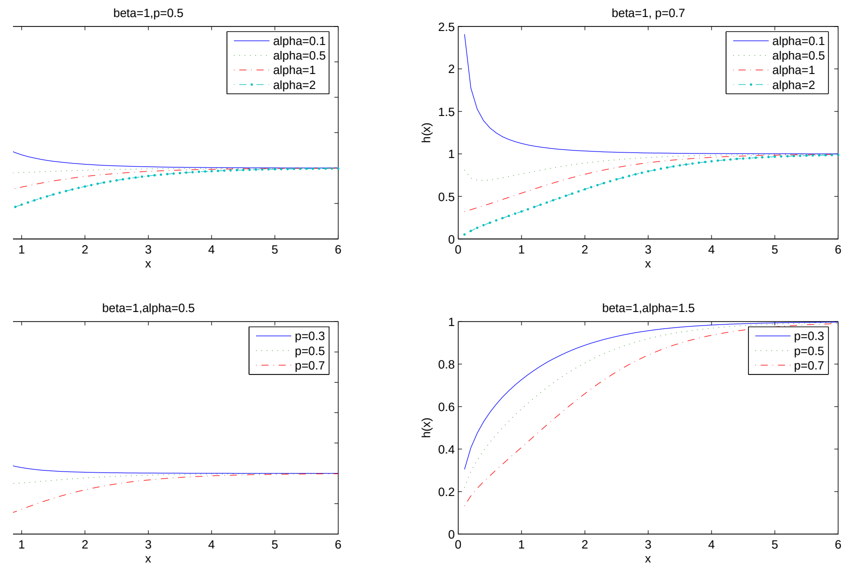

Figure 2: Hazard rate function of the max EEG distribution.

$$E(X^r) = (1-p)\frac{\alpha r!}{\beta^r}\sum_{k=0}^{\infty}(-p)^k(k+\alpha)\sum_{i=0}^{\infty}(-1)^i\binom{k+\alpha-1}{i}\frac{1}{(i+1)^{r+1}},$$

The first two moments are:

$$E(X) = \frac{(1-p)}{\beta}\sum_{k=0}^{\infty}(-p)^k(k+\alpha)\sum_{i=0}^{\infty}(-1)^i\binom{k+\alpha-1}{i}\frac{1}{(i+1)^2}$$

and

$$E(X^2) = \frac{2(1-p)}{\beta^2}\sum_{k=0}^{\infty}(-p)^k(k+\alpha)\sum_{i=0}^{\infty}(-1)^i\binom{k+\alpha-1}{i}\frac{1}{(i+1)^3}.$$

6 Rényi Entropy

An entropy is a measure of variation of the uncertainty. The Rényi entropy of a random variable with probability density function $f(.)$ is defined as

$$I_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^{\infty} f^{\gamma}(x)dx$$

for $\gamma > 0$ and $\gamma \neq 1$.

Here, we derive expressions for the Rényi entropies.

$$\begin{aligned}
\int_0^{\infty} f^{\gamma}(x)dx &= \int_0^{\infty}(\alpha\beta(1-p))^{\gamma}\exp(-\beta x\gamma)\frac{(1-\exp(-\beta x))^{\gamma\alpha-\gamma}}{(1-p(1-\exp(-\beta x))^{\alpha})^{2\gamma}}dx \\
&= (\alpha\beta(1-p))^{\gamma}\sum_{k=0}^{\infty}\binom{-2\gamma}{k}p^k\int_0^{\infty}\exp(-\beta x\gamma)(1-\exp(-\beta x))^{\gamma\alpha-\gamma+\alpha k}dx \\
&= (\alpha(1-p))^{\gamma}\beta^{\gamma-1}\sum_{k=0}^{\infty}\binom{-2\gamma}{k}p^k\int_0^{\infty} y^{\gamma-1}(1-y)^{\gamma\alpha-\gamma+\alpha k}dy \\
&= (\alpha(1-p))^{\gamma}\beta^{\gamma-1}\sum_{k=0}^{\infty}\binom{-2\gamma}{k}p^k B(\gamma,\gamma\alpha-\gamma+\alpha k+1)
\end{aligned}$$

$$I_R(\gamma) = \frac{\gamma}{1-\gamma}\log\Big(\alpha(1-p)\Big)+\log\beta + \frac{1}{\gamma-1}\log\Big(\sum_{k=0}^{\infty}\binom{-2\gamma}{k}p^k B(\gamma,\gamma\alpha-\gamma+\alpha k+1)\Big)$$

7 Estimation of the parameters

7.1 Estimation by maximum likelihood

In this section, we want to obtain the maximum likelihood estimation of parameters. The log-likelihood function based on the observed sample size of n, $y_{obs} = (x_i; i = 1, 2, \ldots, n)$ from (2) is given by

$$\begin{aligned}\ell &= n\log\Big(\alpha\beta(1-p)\Big)-\beta\sum_{i=1}^{n}x_i + \sum_{i=1}^{n}(\alpha-1)\log\Big(1-\exp(-\beta x)\Big)\\ &\quad -2\sum_{i=1}^{n}\log\Big(1-p(1-\exp(-\beta x))^{\alpha}\Big)\end{aligned}$$

Differentiating with respect to θ and equating to zero we obtain the gradients as

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha}+\sum_{i=1}^{n}\log\Big(1-\exp(-\beta x_i)\Big)+2p\sum_{i=1}^{n}\frac{\Big(1-\exp(-\beta x_i)\Big)^{\alpha}\log\Big(1-\exp(-\beta x_i)\Big)}{1-p\Big(1-\exp(-\beta x_i)\Big)^{\alpha}} \tag{8}$$

$$\frac{\partial\ell}{\partial\beta} = \frac{n}{\beta}-\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}(\alpha-1)\frac{x_i\exp(-\beta x_i)}{1-\exp(-\beta x_i)}+2\alpha p\sum_{i=1}^{n}\frac{x_i\exp(-\beta x_i)\Big(1-\exp(-\beta x_i)\Big)^{\alpha-1}}{1-p\Big(1-\exp(-\beta x_i)\Big)^{\alpha}} \tag{9}$$

$$\frac{\partial\ell}{\partial p} = \frac{-n}{1-p}+2\sum_{i=1}^{n}\frac{\Big(1-\exp(-\beta x_i)\Big)^{\alpha}}{1-p\Big(1-\exp(-\beta x_i)\Big)^{\alpha}} \tag{10}$$

To achieve estimations via ML method, it is not easy to solve the equations $\frac{\partial\ell}{\partial\alpha}$, $\frac{\partial\ell}{\partial\beta}$ and $\frac{\partial\ell}{\partial p}$, directly. In the following, theorems 1, 2 and 3 express the conditions for the existence and uniqueness of the MLE, when the other parameters are given or known.

7.2 EM algorithm

An alternative method is the EM algorithm. It is a very powerful tool in handling the incomplete data problem ([3]; [7]). It is an iterative method by repeatedly replacing the missing data with estimated values and updating the parameters. It is especially useful if the complete data set is easy to analyze. Recently, EM algorithm has been used by several authors, see [2], [1], [8], [6] and [9].

To start the algorithm, hypothetical complete-data distribution is defined with density function $f(x, z; \alpha, \beta, p) = \alpha\beta z p^{z-1}(1-p)\exp(-\beta x)(1-\exp(-\beta x))^{\alpha z-1}$, $z = 1, 2, \ldots$; $\alpha, \beta, x > 0$ and $0 \le p \le 1$. Thus, it is straightforward to verify that the Estep of an EM cycle requires the computation of the conditional expectation of $(Z|X; \alpha^{(h)}, \beta^{(h)}, p^{(h)})$, where $(\alpha^{(h)}, \beta^{(h)}, p^{(h)})$ is the current estimate of (α, β, p). Using,
$p(z|x) = zp^{z-1}\Big(1-\exp(-\beta x)\Big)^{\alpha z-\alpha}\Big(1-p(1-\exp(-\beta x))^{\alpha}\Big)^2$, the following found to be

$$E(z|x, \alpha^{(h)}, \beta^{(h)}, p^{(h)}) = \frac{\Big(1+p^{(h)}(1-\exp(-\beta^{(h)}x))^{\alpha^{(h)}}\Big)}{\Big(1-p^{(h)}(1-\exp(-\beta^{(h)}x))^{\alpha^{(h)}}\Big)}.$$

The EM cycle is completed with M-step, which is complete data maximum likelihood over (α, β, p), with the missing Zs replaced by their conditional expectations $E(z|x, \alpha^{(h)}, \beta^{(h)}, p^{(h)})$ (Adamidis and Loukas, 1998). Thus, an EM iteration is given by

$$\alpha^{(h+1)} = \frac{-n}{\sum_{i=1}^{n} z_i \log\Big(1 - \exp(-\beta^{(h)} x_i)\Big)} \tag{11}$$

$$\frac{n}{\beta^{(h+1)}} + \sum_{i=1}^{n} \frac{x_i \exp(-\beta^{(h+1)} x_i)\Big(\alpha^{(h)} z_i - 1\Big)}{\Big(1 - \exp(-\beta^{(h+1)} x_i)\Big)} = \sum_{i=1}^{n} x_i \tag{12}$$

and

$$p^{(h+1)} = \frac{\sum_{i=1}^{n} z_i - n}{\sum_{i=1}^{n} z_i} \tag{13}$$

where

$$z_i = \frac{\Big(1 + p^{(h)}\big(1 - \exp(-\beta^{(h)} x)\big)^{\alpha^{(h)}}\Big)}{\Big(1 - p^{(h)}\big(1 - \exp(-\beta^{(h)} x)\big)^{\alpha^{(h)}}\Big)}$$

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Asymptotic Interval Estimation for the Stress-Strength Reliability in Generalized Logistic Distribution

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Abstract

This paper deals with the asymptotic interval estimation of $P(Y < X)$ when X and Y are two independent generalized Logistic distributions. The maximum likelihood estimator and its asymptotic distribution are obtained. Based on the asymptotic distribution , an asymptotic confidence interval for $P(Y < X)$ is constructed. Finally a numerical example is given to illustrate the results.

Keywords and Phrases: Generalized Logistic distribution, Maximum likelihood estimator, Asymptotic distribution

AMS Subject Classification 2000: primary 62F10; secondary 62F15.

1 Introduction

In the context of reliability, the stress-strength model describes the life of a component which has a random strength X and is subjected to random stress Y. Then $R = P(Y < X)$ can be considered as a measure of system performance. The system becomes out of control if the system stress exceeds its strength. Since R represents a relation between the stress and strength of a system, it is popularly known as the stress-strength parameter of that system. For more application of R, see Kotz et al. (2003). Many authors have studied the stress-strength parameter R. Among them, Awad et al. (1981), Gupta and Gupta (1990), Ahmad et al. (1997), Kundu and Gupta (2005) and Raqab et al. (2008).

Balakrishnan and Leung (1988) defined the generalized logistic (GL) distribution as one of three generalized forms of the standard logistic distribution. The GL distribution has received additional attention in estimating its parameters for practical usage. See, for example, Balakrishnan (1991) and Asgharzadeh (2006). For $\alpha > 0$ and $\lambda > 0$, the two-parameter GL distribution has the cumulative distribution function (cdf)

$$F(x;\alpha,\lambda) = (1+e^{-\lambda x})^{-\alpha}, \qquad -\infty < x < \infty \tag{1.1}$$

and has the probability density function (pdf)

$$f(x;\alpha,\lambda) = \alpha\lambda e^{-\lambda x}(1+e^{-\lambda x})^{-\alpha-1}. \qquad -\infty < x < \infty \tag{1.2}$$

Here α and λ are the shape and scale parameters, respectively. The two-parameter GL distribution will be denoted by $GL(\alpha,\lambda)$. The density in (1.2) has been obtained by compounding an extreme value distribution with a gamma distribution. It is observed by Balakrishnan and Leung (1988) that this distribution is skewed and its kurtosis coefficient is greater than that of the logistic distribution.

In this paper, we consider the problem of estimating $R = P(Y < X)$, under the assumption that $X \sim GL(\alpha,\lambda)$, $Y \sim GL(\beta,\lambda)$, and X and Y are independently distributed. Then it can be easily seen that

$$R = P(Y < X) = \frac{\alpha}{\alpha+\beta}. \tag{1.3}$$

2 Maximum Likelihood Estimator of R

To compute the maximum likelihood estimator (MLE) of R, first we obtain the MLE of α, β and λ. Suppose $X_1, X_2, \cdots, X_n$ is a random sample from $GL(\alpha,\lambda)$ and $Y_1, Y_2, \cdots, Y_m$ is a random sample from $GL(\beta,\lambda)$. Therefore the log-likelihood function of α, β and λ is given by

$$\begin{aligned} l(\alpha,\beta,\lambda) &= n\ln(\alpha) + m\ln(\beta) + (n+m)\ln(\lambda) - \lambda\left[\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j\right] \\ &- (\alpha+1)S_1(\mathbf{x},\lambda) - (\beta+1)S_2(\mathbf{y},\lambda) \end{aligned}$$

where

$$S_1(\mathbf{x},\lambda) = \sum_{i=1}^{n} \ln(1+e^{-\lambda x_i}) \quad , \quad S_2(\mathbf{y},\lambda) = \sum_{j=1}^{m} \ln(1+e^{-\lambda y_j}) \tag{2.2}$$

.

The MLEs of α, β and λ, say $\widehat{\alpha}$, $\widehat{\beta}$ and $\widehat{\lambda}$ respectively, can be obtained as the solution of

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - S_1(\mathbf{x},\lambda) = 0, \tag{2.3}$$

$$\frac{\partial l}{\partial \beta} = \frac{m}{\beta} - S_2(\mathbf{y},\lambda) = 0, \tag{2.4}$$

$$\begin{aligned} \frac{\partial l}{\partial \lambda} &= \frac{n+m}{\lambda} - \left[\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j\right] + (\alpha+1)\sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{1+e^{-\lambda x_i}} \\ &+ (\beta+1)\sum_{j=1}^{m} \frac{y_j e^{-\lambda y_j}}{1+e^{-\lambda y_j}} = 0. \end{aligned} \tag{2.5}$$

From (2.3) and (2.4), we obtain

$$\widehat{\alpha}(\lambda) = \frac{n}{S_1(\mathbf{x},\lambda)} \quad , \quad \widehat{\beta}(\lambda) = \frac{m}{S_2(\mathbf{y},\lambda)}. \tag{2.6}$$

Putting the values of $\widehat{\alpha}(\lambda)$ and $\widehat{\beta}(\lambda)$ into (2.5), $\widehat{\lambda}$ can be obtained as a fixed point solution of the following equation

$$g(\lambda) = \lambda \tag{2.7}$$

where

$$g(\lambda) = \frac{n+m}{\sum_{i=1}^{n} x_i + \sum_{j=1}^{m} y_j - (\widehat{\alpha}(\lambda)+1)\sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{1+e^{-\lambda x_i}} - (\widehat{\beta}(\lambda)+1)\sum_{j=1}^{m} \frac{y_j e^{-\lambda y_j}}{1+e^{-\lambda y_j}}}$$

A simple iterative procedure $g(\lambda^{(j+1)}) = \lambda^{(j+1)}$ where $\lambda^{(j)}$ is the jth iterate, can be used to find the solution of (2.7). Once we obtain $\widehat{\lambda}_{ML}$, the MLE of α and β, can be deduced from (2.3) and (2.4) as $\widehat{\alpha}_{ML} = \widehat{\alpha}(\widehat{\lambda}_{ML})$ and $\widehat{\beta}_{ML} = \widehat{\beta}(\widehat{\lambda}_{ML})$. Therefore, we compute the MLE of R as

$$\widehat{R}_{ML} = \frac{\widehat{\alpha}_{ML}}{\widehat{\alpha}_{ML} + \widehat{\beta}_{ML}}. \tag{2.8}$$

3 Asymptotic Interval Estimation

In this section, first we obtain the asymptotic distribution of $\widehat{\theta} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\lambda})$ and then derive the asymptotic distribution of $\widehat{R}$. Based on the asymptotic distribution of $\widehat{R}$, we obtain the asymptotic confidence interval of R. Let us denote the Fisher information matrix of $\theta = (\alpha, \beta, \lambda)$ as $\mathbf{J}(\theta) = E(\mathbf{I}; \theta)$ where $\mathbf{I} = (I_{ij}(\theta))$ for $i, j = 1, 2, 3$. Therefore,

$$\mathbf{I}(\theta) = -\begin{pmatrix} \frac{\partial^2 l}{\partial \alpha^2} & \frac{\partial^2 l}{\partial \alpha \partial \beta} & \frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 l}{\partial \beta \partial \alpha} & \frac{\partial^2 l}{\partial \beta^2} & \frac{\partial^2 l}{\partial \beta \partial \lambda} \\ \frac{\partial^2 l}{\partial \lambda \partial \alpha} & \frac{\partial^2 l}{\partial \lambda \partial \beta} & \frac{\partial^2 l}{\partial \lambda^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}$$

It is easy to see that

$$\begin{aligned}
I_{11} &= \frac{n}{\alpha^2}, \\
I_{12} &= I_{21} = 0, \\
I_{13} &= I_{31} = -\sum_{i=1}^{n} \frac{x_i e^{-\lambda x_i}}{1 + e^{-\lambda x_i}} \\
I_{22} &= \frac{m}{\beta^2} \\
I_{23} &= I_{32} = -\sum_{j=1}^{m} \frac{y_j e^{-\lambda y_j}}{1 + e^{-\lambda y_j}} \\
I_{33} &= \frac{n+m}{\lambda^2} + (\alpha+1) \sum_{i=1}^{n} \frac{x_i^2 e^{-\lambda x_i}}{(1 + e^{-\lambda x_i})^2} + (\beta+1) \sum_{j=1}^{m} \frac{y_j^2 e^{-\lambda y_j}}{(1 + e^{-\lambda y_j})^2}
\end{aligned}$$

We have

$$\begin{aligned}
J_{11} &= \frac{n}{\alpha^2}, \\
J_{12} &= J_{21} = 0, \\
J_{13} &= J_{31} = -\frac{n}{\lambda(\alpha+1)}[\Psi(\alpha) + \gamma - 1] \\
J_{22} &= \frac{m}{\beta^2} \\
J_{23} &= J_{32} = -\frac{m}{\lambda(\beta+1)}[\Psi(\beta) + \gamma - 1] \\
J_{33} &= \frac{n+m}{\lambda^2} + \frac{n}{\lambda^2(\alpha+2)} \\
&\times \left[\alpha[\Psi'(\alpha) + \Psi^2(\alpha)] + 2\left[(\alpha(\gamma-1)+1)\Psi(\alpha) - (\gamma(\alpha-1)+1)\right] + \alpha[\gamma^2 + \frac{\pi^2}{6}]\right] \\
&+ \frac{m}{\lambda^2(\beta+2)} \\
&\times \left[\beta[\Psi'(\beta) + \Psi^2(\beta)] + 2\left[(\beta(\gamma-1)+1)\Psi(\beta) - (\gamma(\beta-1)+1)\right] + \beta[\gamma^2 + \frac{\pi^2}{6}]\right]
\end{aligned}$$

where $\Psi(t) = \frac{d}{dt}\ln(\Gamma(t))$, $\Psi'(t) = \frac{d}{dt}\Psi(t)$ and $\gamma = -\Psi(1) = 0.5772$.

Theorem 1: As $n \to \infty$, $m \to \infty$ and $\frac{n}{m} \to p$, then

$$\left(\sqrt{n}(\widehat{\alpha} - \alpha),\ \sqrt{m}(\widehat{\beta} - \beta),\ \sqrt{n}(\widehat{\lambda} - \lambda)\right) \xrightarrow{d} N_3(0, \mathbf{A}^{-1}(\alpha, \beta, \lambda))$$

where

$$\mathbf{A}(\alpha, \beta, \lambda) = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

and

$$a_{11} = \lim_{n,m\to\infty} \frac{J_{11}}{n} = \frac{1}{\alpha^2}, \qquad a_{13} = a_{31} = \lim_{n,m\to\infty} \frac{J_{13}}{n} = -\frac{\Psi(\alpha)+\gamma-1}{\lambda(\alpha+1)}$$

$$a_{22} = \lim_{n,m\to\infty} \frac{J_{22}}{m} = \frac{1}{\beta^2}, \qquad a_{23} = a_{32} = \lim_{n,m\to\infty} \frac{\sqrt{p}}{n} J_{23} = -\frac{\Psi(\beta)+\gamma-1}{\lambda(\beta+1)\sqrt{p}}$$

$$a_{33} = \lim_{n,m\to\infty} \frac{J_{33}}{n} = \frac{p+1}{p\lambda^2} + \frac{1}{\lambda^2(\alpha+2)}$$
$$\times \left[\alpha[\Psi'(\alpha)+\Psi^2(\alpha)] + 2\left[(\alpha(\gamma-1)+1)\Psi(\alpha) - (\gamma(\alpha-1)+1)\right] + \alpha[\gamma^2+\frac{\pi^2}{6}]\right]$$
$$+\frac{1}{p\lambda^2(\beta+2)}$$
$$\times \left[\beta[\Psi'(\beta)+\Psi^2(\beta)] + 2\left[(\beta(\gamma-1)+1)\Psi(\beta) - (\gamma(\beta-1)+1)\right] + \beta[\gamma^2+\frac{\pi^2}{6}]\right]$$

Proof : The proof follows by expanding the derivative of the log-likelihood function using Taylor series, and using the Central limit theorem. Now we have the main result:
Theorem 2: As $n \to \infty$, $m \to \infty$ and $\frac{n}{m} \to p$, then

$$\sqrt{n}(\widehat{R} - R) \to N(0, B) \tag{2.10}$$

where B=$\mathbf{b^t A^{-1} b}$, and

$$\mathbf{b} = \begin{pmatrix} \frac{\partial R}{\partial \alpha} \\ \frac{\partial R}{\partial \beta} \\ \frac{\partial R}{\partial \lambda} \end{pmatrix} = \frac{1}{(\alpha+\beta)^2} \begin{pmatrix} \beta \\ -\alpha \\ 0 \end{pmatrix} \quad , \quad \mathbf{A^{-1}} = \frac{1}{u} \begin{pmatrix} a_{22}a_{33} - a_{23}^2 & a_{13}a_{32} & -a_{13}a_{22} \\ a_{23}a_{31} & a_{11}a_{33} - a_{13}^2 & -a_{11}a_{23} \\ -a_{22}a_{31} & -a_{11}a_{32} & a_{11}a_{22} \end{pmatrix}$$

and

$$u = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{13}a_{22}a_{31}.$$

Proof: It follows from Theorem 1.

It's easy to show that

$$B = \frac{1}{u(\alpha+\beta)^4}\left[\beta^2(a_{22}a_{33} - a_{23}^2) - 2\alpha\beta(a_{13}a_{23}) + \alpha^2(a_{11}a_{33} - a_{13}^2)\right]$$

Theorem 2 can be used to construct the asymptotic confidence interval of R. To compute the confidence interval of R, the variance B needs to be estimated. We recommend using the empirical Fisher information matrix, and the MLE estimates of α, β and λ to estimate B, which is very convenient. Now, we can obtain the $100(1-\gamma)\%$ confidence interval for R by using asymptotic distributions of the MLE as folows:

$$(\widehat{R} - z_{1-\frac{\gamma}{2}}\frac{\sqrt{\widehat{B}}}{\sqrt{n}}, \widehat{R} + z_{1-\frac{\gamma}{2}}\frac{\sqrt{\widehat{B}}}{\sqrt{n}}).$$

where z_γ is 100γth percentile of $N(0,1)$.

4 Data Analysis

In this section, we present a complete analysis of simulated data. The data has been generated using $n = m = 15$ and $\alpha = 1$, $\beta = 2$ and $\lambda = 1$. Therefore, $R = 0.333$. The X values are

```
-1.927  -1.697  -1.296  -1.188  -1.100  -1.004  -0.433  -0.427
-0.319  -0.273  -0.155   0.011   0.284   2.059   3.147
```

and the corresponding Y values are

```
-1.046  -0.940  -0.881  -0.661  -0.484  -0.178   0.104  0.185
 1.020   1.255   1.283   1.685   2.025   2.269   2.589
```

The ML estimation of R become 0.363, and the corresponding 95% confidence intervals become (0.195,0.530).

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Bayesian inference for geometric distribution under a simple step-stress model

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Abstract

In some situations in reliability and survival analysis, the life times of the units in an experiment depend on the number of times the units are switched on and off or the number of shocks they receive. On the other hand, the experiment may not terminate on an adequate time under the normal conditions. This paper proposes a Bayesian inference model for a simple step-stress model with Type-I censored sample. Assuming a cumulative exposure model with lifetimes being geometric distributed, the problems of point and interval estimation of studied in the Bayesian approach. Finally, an example is presented to illustrate the proposed procedure in this paper.

Keywords and phrases: Bayesian confidence interval, Beta prior distribution, Accelerated testing, Order statistics.

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1 Introduction

There are situations in reliability and survival analysis for which the experiment may not terminate on an adequate time under the normal conditions. In such situations, accelerated life-testing experiments have been offered to obtain adequate life data. See, for example, Nelson (1990) and Bagdonavicius and Nikulin (2002). Step-stress accelerated life testing (SSALT) is a special class of accelerated life-testing for which the stress levels of the experiment change at some pre-specified times. Balakrishnan *et al.* (2009) derived exact inference for simple step-stress model from the exponential distribution when there is time constraint on the duration of the experiment. See also, Balakrishnan and Xie (2007, a) and Balakrishnan and Han (2008). DeGroot and Goel (1979) proposed a Bayesian inference model for SSALT and an criterion optimality for simple SSALT in the framework of Bayesian decision theory. See also, Van Dorp et al. (1996) and Van Dorp and Mazzuchi (2004, 2005) and Erto, and Giorgio (2002). The classical approach treats parameters of life distribution as fixed but unknown constants but Bayes approach considers them as random variables whit a prior distributions for these parameters. Prior distributions are constructed by existing information or subjective judgments.
In some situations, the life times of the units in an experiment depend on the number of times the units are switched on and off or the number of shocks they receive. Let w be the number of switch on and off or shocks the units receive until they fail, so, w is considered as the associated failure time. Here, the life-testing experiment are investigated in a discrete set up. See, Nagaraja (1992) for more details about the results on order statistics of a random sample taken from a discrete population. Censored samples in discrete set up have been studied by some authors. See, for example, Rezaei and Arghami (2002), Davarzani and Parsian (2011) and Balakrishnan *et al.* (2011). This paper proposes a Bayesian inference model for a simple SSALT having only one change between two stress levels, when Type-I censoring is used. We assume that the failure times at each stress level follow a geometric distribution.

The rest of the paper is as follows: In Section 2 some preliminaries are presented. In Section 3, the Byesian estimation of the parameters of the geometric distribution is investigated for a simple step-stress model with Type-I censored sample. Finally, in Section 4, an example is given to illustrate the results of the paper.

2 Preliminaries

Consider a simple step-stress scheme with only two stress levels S_1 and S_2 and assume that the associated distributions at levels S_1 and S_2 are geometric with successive probabilities p_1 and p_2, respectively. The probability mass function (pmf) and cumulative distribution function (cdf) are given by

$$\mathrm{P}_j(X_j = x) = p_j q_j^{x-1}, \quad x = 1, 2, \ldots,$$

$$F_j(x; p_j) = 1 - q_j^x, \quad x = 1, 2, \ldots \tag{1}$$

The parameters of interest in this paper are:(i) the successive probability at level S_j, i.e.,p_j, ($j = 1, 2$), (ii) the mathematical expectation at level S_j,i.e., $\mu_j = \frac{1}{p_j}$, (iii) the survival function of the level S_j at $x_0 : \bar{F}_j(x_0) = q_j^{x_0}$.
Suppose that the normal conditions (level S_1) of an experiment change to level S_2 at point w_1. Therefore, using (1), the cumulative exposure distribution (ced) $G(x)$ is

$$\begin{aligned} G(x) &= \begin{cases} G_1(x) = F_1(x; p_1), & x = 1, 2, \ldots, w_1, \\ G_2(x) = F_2\big(x - (1 - \log q_1 / \log q_2) w_1; p_2\big), & x = w_1 + 1, w_1 + 2, \ldots, \end{cases} \\ &= \begin{cases} G_1(x) = 1 - q_1^x, & x = 1, 2, \ldots, w_1, \\ G_2(x) = 1 - q_1^{w_1} q_2^{x - w_1}, & x = w_1 + 1, w_1 + 2, \ldots \end{cases} \end{aligned} \tag{2}$$

and the corresponding pmf $g(x)$ is as follows

$$g(x) = \begin{cases} g_1(x) = p_1 q_1^{x-1}, & x = 1, 2, \ldots, w_1, \\ g_2(x) = p_2 q_1^{w_1} q_2^{x-(w_1+1)}, & x = w_1 + 1, w_1 + 2, \ldots. \end{cases}$$

We now introduce some notations will be used throughout the paper: $X_{i:n}$ denotes the ith smallest order statistics in a sample of size n from the geometric distribution. N_1 is the number of observations that are less than or equal to w_1 and N_2 denotes the number of data points that are less than or equal to w_2 and greater than w_1, for which $N_1 + N_2 \leq n$. Using these notation, we will observe the following data set under Type-I censoring scheme:

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{N_1:n} \leq w_1 < X_{N_1+1:n} \leq \cdots \leq X_{N_1+N_2:n} \leq w_2. \tag{3}$$

Notice that in the special case of $N_1 + N_2 = n$, the complete sample is observed. To study the estimation problem of the parameter of interest based on the data set in (3), we need to obtain the joint distribution of $X_{1:n}, \ldots, X_{N_1+N_2:n}$. In order to provide explicit expression for the joint distribution of discrete-order statistics, it is necessary to use the 'tie- run" technique which is defined by Gan and Bain (1995) regarding the number and lengths of runs of tied observations. A subchain $t_{i_1} \leq t_{i_2} \leq \cdots \leq t_{i_n}$ of real numbers is said to have r tie-runs ($1 \leq r \leq n$) with length z_k ($1 \leq k \leq r$) for the kth one, if

$$t_{i_1} = \cdots = t_{i_{z_1}} < t_{i_{z_1+1}} = \cdots = t_{i_{z_1+z_2}} < \cdots < t_{i_{n-z_r+1}} = \cdots = t_{i_n},$$

with $\sum_{k=1}^{r} z_k = n$. Let $X_1, \ldots, X_n$ be iid discrete random variables from the ced in (2). Using the concept of tie-run, given p_1 and p_2, the joint pmf of $X_{1:n}, \cdots, X_{N_1+N_2:n}, N_1, N_2$ is as follows

$$\begin{aligned} L(p_1, p_2) &= \frac{n!}{(n - n_1 - n_2)!} \left(\prod_{j=1}^{r} z_j! \right)^{-1} \prod_{i=1}^{n_1} g_1(x_{i:n}) \\ &\times \prod_{i=n_1+1}^{n_1+n_2} g_2(x_{i:n}) \big(1 - G_2(w_2)\big)^{n-n_1-n_2} \\ &= \frac{n!}{(n - n_1 - n_2)!} \left(\prod_{j=1}^{r} z_j! \right)^{-1} p_1^{n_1} q_1^{d_1} p_2^{n_2} q_2^{d_2}, \end{aligned} \tag{4}$$

where r is equal to the number of tie-runs with length z_k for the jth one, d_1 and d_2 are the observed values of , D_1 and D_2, respectively, where

$$D_1 = \sum_{i=1}^{N_1} X_{i:n} - N_1 + w_1(n - N_1), \tag{5}$$

$$D_2 = \sum_{i=N_1+1}^{N_1+N_2} X_{i:n} - (w_1+1)N_2 + (w_2 - w_1)(n - N_1 - N_2). \tag{6}$$

3 Baysian estimation

In this section we present and illustrate the methodology for obtaining the Bayes estimators.Toward this end, we assume that the parameters p_1 and p_2 behave as independent random variables. Also, suppose the random variable p_j has Beta prior distribution with parameters α_j and β_j $(j = 1, 2)$. That is, the prior density function of $p_j,\ j = 1, 2$, takes the following form

$$\pi_j(p_j) = \frac{1}{\beta(\alpha_j, \beta_j)} p_j^{\alpha_j - 1}(1 - p_j)^{\beta_j - 1}, \quad 0 < p_j < 1. \tag{7}$$

Therefore, by performing some algebraic calculations, it con be shown that, the joint posterior pdf of p_1 and p_2 is

$$\pi(p_1, p_2|data) = \frac{p_1^{a_1-1}(1-p_1)^{b_1-1} p_2^{a_2-1}(1-p_2)^{b_2-1}}{\prod_{j=1}^{2} B(a_j, b_j)}, \tag{8}$$

where $a_j = N_j + \alpha_j$. and $b_j = D_j + \beta_j$ $(j = 1, 2)$, for which D_1 and D_2 are as defined in (5) and (6), respectively. Using (8), the marginal posterior of p_j is

$$\pi_j(p_j|data) = \frac{p_j^{a_j-1}(1-p_j)^{b_j-1}}{B(a_j, b_j)}, \quad j = 1, 2. \tag{9}$$

3.1 Bayesian point estimation

To proceed the problem of Bayes estimation, we use the squared error loss (SEL) function. Let $\hat{\theta}$ be any estimator for θ, then the SEL function is defined by $L(\hat{\theta} - \theta) = (\hat{\theta} - \theta)^2$. Using the SEL function, the Bayes estimate of the unknown parameter is simply the mean of the posterior distribution. It can be shown that the Bayes risk is the variance of the posterior distribution. In the following results, the Bayes estimators for the parameters of interest in this paper are presented. Under the of SEL function, we have
(i) The Bayes estimator for $p_j (j = 1, 2)$ is given by

$$\hat{p}_j = \frac{a_j}{a_j + b_j}; \tag{10}$$

(ii) The Bayes risk associated with $\hat{p}_j$ say $R_{\hat{p}_j}$, is

$$R_{\hat{p}_j} = \frac{a_j b_j}{(a_j + b_j + 1)(a_j + b_j)^2}.$$

the Bayes estimator for $\mu_j,\ j = 1, 2$ is

$$\hat{\mu}_j = \frac{a_j + b_j - 1}{a_j - 1}, \tag{11}$$

Moreover the Bayes risk associated with $\hat{\mu}_j$ say $R_{\hat{\mu}_j}$, is

$$R_{\hat{\mu}_j} = \frac{(a_j + b_j - 2)(a_j + b_j - 1)}{(a_j - 2)(a_j - 1)} - \left(\frac{a_j + b_j - 1}{a_j - 1}\right)^2.$$

For estimating the survival function of the level S_j at x_0, we have (i) The Bayes estimator for $\bar{F}_j(x_0)(j = 1, 2)$ is

$$\hat{\bar{F}}_j(x_0) = \frac{b_j^{[x_0]}}{(a_j + b_j)^{[x_0]}}; \tag{12}$$

where

$$\begin{aligned} b_j^{[x_0]} &= (b_j + x_0 - 1)(b_j + x_0 - 2) \ldots (b_j + 1)\ b_j, \\ (a_j + b_j)^{[x_0]} &= (a_j + b_j + x_0 - 1)(a_j + b_j + x_0 - 2) \ldots (b_j + a_j), \end{aligned}$$

(ii) The Bayes risk associated with $\hat{\bar{F}}_j(x_0)$ say $R_{\hat{\bar{F}}_j(x_0)}$, is

$$R_{\hat{\bar{F}}_j(x_0)} = \frac{b_j^{[2x_0]}}{(a_j + b_j)^{[2x_0]}} - \left\{ \frac{b_j^{[x_0]}}{(a_j + b_j)^{[x_0]}} \right\}^2;$$

where

$$\begin{aligned} b_j^{[2x_0]} &= (b_j + 2x_0 - 1)(b_j + 2x_0 - 2) \ldots (b_j + 1)\ b_j, \\ (a_j + b_j)^{[2x_0]} &= (a_j + b_j + 2x_0 - 1)(a_j + b_j + 2x_0 - 2) \ldots (b_j + a_j). \end{aligned}$$

3.2 Bayesian interval estimation

Once the posterior probability density function $h(\theta|data)$ of the unknown parameter θ is derived,the interval is $100(1 - \alpha)\%$ Bayesian confidence interval for θ, is

$$P(L \leq \theta \leq U|data) = 1 - \alpha. \tag{13}$$

Using (9) and (13), a $100(1 - \alpha)\%$ Bayesian confidence intervals for $p_j(j = 1, 2)$, say (L_j, U_j), can be derived by solving the following two equations

$$\frac{\alpha}{2} = \int_0^{L_j} \frac{p_j^{a_j - 1}(1 - p_j)^{b_j - 1}}{B(a_j, b_j)} dp_j, \quad \frac{\alpha}{2} = \int_{U_j}^1 \frac{p_j^{a_j - 1}(1 - p_j)^{b_j - 1}}{B(a_j, b_j)} dp_j. \tag{14}$$

Let (L_j, U_j) be a $100(1 - \alpha)\%$ confidence interval for θ_j and $S(\cdot)$ be any increasing function, then $\{S(L_j), S(U_j)\}$ is a $100(1 - \alpha)\%$ confidence interval for $S(\cdot)$, and if $S(\cdot)$ be any decreasing function then $\{S(U_j), S(L_j)\}$ is a $100(1 - \alpha)\%$ confidence interval for $S(\theta_j)$. Therefor, using (14), one may construct a confidence interval for other parameters of interest in the paper.

4 Illustrative example

To illustrate the proposed procedure in this paper, we consider a numerical example. Assuming $w_1 = 15$, a random ample of size 30 has been generated from ced in (2) with $p_1 = 0.015$ and $p_2 = 0.056$. The results are presented in Table 1.

Table 1. Generated sample of size 30 from ced in (2) with $w_1 = 15$, $p_1 = 0.015$ and $p_2 = 0.056$.

Parameter	Failure times											
p_1	4	7	9	13	15							
p_2	16	17	18	18	19	19	20	20	21	23	24	25
	26	28	29	31	32	35	36	40	40	43	48	58

Using the data in Table 1, we would obtain $N_1 = 5$ is the number of units that fail at stress level s_0. To investigate the variety of inference, we use different choices of w_2. From Table 1, for $w_2 = 20, 28$ and 35, the number of units that fail at stress level s_1 is given by $N_2 = 8, 14$ and 18, respectively.

The values of the point estimation for p_j, μ_j and $\bar{F}_j(9)$ $(j = 1, 2)$ have been obtained using (10), (11) and (12), respectively. Similar results for the interval estimation have been derived using (14). Toward this end, we consider the beta prior distributions with parameters (0.43, 23.4) and (2.17, 38.6) for p_1 and p_2, respectively. The results are summarized in Table 2.

Table 2. Values of Bayesian point and interval estimation for the parameters of interest based on the data in Table 1 for $w_1 = 15$ and some choices of w_2.

	$w_2 = 20$		$w_2 = 28$		$w_2 = 35$	
	point	interval	point	interval	point	interval
p_1	0.0122	(0.0042, 0.0242)				
p_2	0.0666	(0.0328, 0.1110)	0.0604	(0.0352, 0.0918)	0.0608	(0.0377, 0.0889)
μ_1	100.64	(41.310, 237.15)				
μ_2	16.5507	(9.0089, 30.441)	17.5854	(10.891, 28.437)	17.2546	(11.253, 26.504)
$\bar{F}_1(9)$	00.8859	(00.8021 0.9627)				
$\bar{F}_2(9)$	00.5117	(00.3468, 0.7404)	0.5417	(0.4203, 0.7246)	0.5384	(0.4328, 0.7074)

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Generalized confidence intervals for the survival function of the exponential distribution

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Abstract

The interval estimation of the survival function of the two-parameter exponential distribution on the basis of the progressively Type-II censored samples is investigated. Toward this end, the concept of the generalized confidence intervals (GCIs) is used and the lower and upper generalized confidence limits (GCLs) are obtained. It will be shown that the coverage probabilities of the GCLs are satisfactory using a simulation study. Finally, some concluding remarks are presented.

Keywords and Phrases: Generalized pivotal quantity, Order statistics, Progressively Type-II censored data

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1 Introduction

It is well-known that the exponential distribution is one of the commonly used models in several areas of statistical practice, including survival and reliability analysis. It is used to model data with a constant failure rate; for more details concerning the exponential model and related topics, one may refer to the book by Balakrishnan and Basu [2]. A random variable X is said to have a two-parameter exponential distribution if its cumulative distribution function (cdf) is

$$F(x;\mu,\sigma) = 1 - e^{-(x-\mu)/\sigma}, \quad x \geq \mu, \quad \sigma > 0, \tag{1}$$

where μ and σ are the location and scale parameters, respectively. A problem of interest in the reliability analysis is to investigate the confidence intervals (CIs) for the survival function at a specified point τ, which for the two-parameter exponential distribution is defined by

$$R(\tau;\mu,\sigma) = e^{-(\tau-\mu)/\sigma}, \quad \tau \geq \mu. \tag{2}$$

Engelhardt and Bain [5] suggested an approximate method based on Type-II censored data. See also, Roy and Mathew [7] and Fernández [6]. In this paper, we study the problem of constructing GCIs for $R(\tau;\mu,\sigma)$ on the basis of the progressively Type-II censored order statistics. The model of progressive Type-II censoring is of importance in the field of reliability and life testing. Suppose n units are simultaneously placed on a lifetime test. At the time of the ith failure, R_i surviving units are randomly censored from the experiment, $1 \leq i \leq m$. Thus, if m failures are observed, then $R_1 + \cdots + R_m$ units are progressively censored; here, $\mathbf{R} = (R_1, \ldots, R_m)$ denotes the progressive censoring scheme. The interested readers may refer to the book by Balakrishnan and Aggarwala [1]. See also, Balakrishnan *et al.* [3] and Burkschat *et al.* [4].

The rest of the paper is as follows: In Section 2, some preliminaries are presented. In Section 3, the GCIs for the survival function of the two-parameter exponential distribution are derived on the basis of the progressively Type-II censored order statistics. In section 4, some concluding remarks are stated.

2 Preliminaries

Let $\{Y_1, \ldots, Y_n\}$ be a random sample of size n from the two-parameter exponential distribution with cdf in (1). Denote the first m progressively Type-II censored order statistics by $Y^{\mathbf{R}}_{1:m:n} \leq \cdots \leq Y^{\mathbf{R}}_{m:m:n}$ $(1 \leq$

$m \leq n$), where $\mathbf{R} = (R_1, \ldots, R_m)$ stands for the corresponding progressive censoring scheme. The likelihood function of the parameters of the two-parameter exponential distribution with cdf in (1) based on the progressively Type-II censored order statistics can be written as

$$L(\mu, \sigma) = c\sigma^{-m} \exp\left\{-\sum_{i=1}^{m}(R_i + 1)\frac{y_i - \mu}{\sigma}\right\},$$

where $c = n(n-R_1-1)\cdots(n-R_1-R_2-...-R_{m-1}-m+1)$ and y_i is the observed value of $Y^{\mathbf{R}}_{i:m:n}$. Assuming $m \geq 2$, the maximum likelihood estimators (MLEs) of μ and σ based on $\mathbf{Y} = \{Y^{\mathbf{R}}_{1:m:n}, ..., Y^{\mathbf{R}}_{m:m:n}\}$ are given by

$$\hat{\mu} = \hat{\mu}\,(\mathbf{Y}) = Y^{\mathbf{R}}_{1:m:n} \text{ and } \hat{\sigma} = \hat{\sigma}\,(\mathbf{Y}) = \frac{1}{m}\sum_{i=2}^{m}(R_i + 1)\,(Y^{\mathbf{R}}_{i:m:n} - Y^{\mathbf{R}}_{1:m:n}). \tag{3}$$

Let us take

$$Z_1 = (\hat{\mu} - \mu)/\sigma \text{ and } Z_2 = \hat{\sigma}/\sigma, \tag{4}$$

where $\hat{\mu}$ and $\hat{\sigma}$ are as defined in (3). It can be shown that $2nZ_1 \sim \chi^2_2$ and independently $2mZ_2 \sim \chi^2_{2(m-1)}$, where χ^2_m stands for a chi-square distribution with m degrees of freedom (see, Balakrishnan *et al.*, [3]).

3 Generalized confidence interval

In this section, we use the concept of the GCI to arrive the exact CIs for $R(\tau; \mu, \sigma)$. Let $\mathbf{X}$ be a random vector whose distribution depends on γ and ξ, a scalar parameter of interest and a nuisance parameter, respectively. Furthermore, let $\mathbf{x}$ denote the observed value of $\mathbf{X}$. The random variable $U(\mathbf{X}; \mathbf{x}, \gamma, \xi)$ is called a generalized pivotal quantity if it satisfies in the following two conditions:

$$\begin{aligned}&(i) \text{ The distribution of } U(\mathbf{X}; \mathbf{x}, \gamma, \xi) \text{ is free of unknown parameters, for fixed } \mathbf{x},\\ &(ii) \text{ The observed value of } U(\mathbf{X}; \mathbf{x}, \gamma, \xi), \text{ i.e., } U(\mathbf{x}; \mathbf{x}, \gamma, \xi), \text{ is equal to } \gamma.\end{aligned} \tag{5}$$

The CIs for γ obtained using the percentiles of $U(\mathbf{X}; \mathbf{x}, \gamma, \xi)$ are referred to as the GCIs. Therefore, the $U_\alpha(\mathbf{x})$ is a $100(1-\alpha)\%$ lower GCL for γ if

$$\mathrm{P}\left(U(\mathbf{X}; \mathbf{x}, \gamma, \xi) \geq U_\alpha(\mathbf{x})\right) = 1 - \alpha. \tag{6}$$

The quantiles $U_\alpha(\mathbf{x})$ and $U_{1-\alpha}(\mathbf{x})$ are the lower and upper $100(1-\alpha)\%$ GCLs for γ, respectively, whereas $[U_{\alpha/2}(\mathbf{x}), U_{1-\alpha/2}(\mathbf{x})]$ is the two-sided equi-tailed $100(1-\alpha)\%$ GCI for γ based on $U(\mathbf{X}; \mathbf{x}, \gamma, \xi)$. Notice that the coverage probability of such a confidence interval could depend on unknown parameters and hence it may not be exactly $1-\alpha$ (see, for details, [8] and [9]).
To construct a GCI for $R(\tau; \mu, \sigma)$, we first look for generalized pivotal quantities for μ and σ, denoted by U_μ and U_σ, respectively, satisfying the properties in (5). That is, the distribution of (U_μ, U_σ) is free of any unknown parameters, and the observed value of (U_μ, U_σ) is (μ, σ). Toward this end, let $\hat{\mu}_0$ and $\hat{\sigma}_0$ denote the observed values of $\hat{\mu}$ and $\hat{\sigma}$, respectively, where $\hat{\mu}$ and $\hat{\sigma}$ are as defined in (3). Consider a choice of U_μ and U_σ as follows

$$U_\mu = \hat{\mu}_0 - \frac{Z_1}{Z_2}\hat{\sigma}_0 \text{ and } U_\sigma = \frac{\hat{\sigma}_0}{Z_2}. \tag{7}$$

Note that a generalized pivotal quantity for any function of μ and σ, say $h(\mu, \sigma)$, is given by $h(U_\mu, U_\sigma)$. Here, the function $h(\mu, \sigma)$ can be quite arbitrary and could be rather complicated. Therefore, using the pivots in (7), a generalized pivotal quantity for $R(\tau; \mu, \sigma)$ in (2) is given by

$$U_R = U_R(\mathbf{Y}; \mathbf{y}, \mu, \sigma) = \exp\{-\frac{\tau - U_\mu}{U_\sigma}\} = \exp\{-Z_1 - \frac{\tau - \hat{\mu}_0}{\hat{\sigma}_0}Z_2\}. \tag{8}$$

Since a confidence limit (CL) for $R(\tau; \mu, \sigma)$ must be restricted to be 1, it is reasonable to use an alternative generalized pivot for $R(\tau; \mu, \sigma)$ as follows

$$U^* = U^*(\mathbf{Y}; \mathbf{y}, \mu, \sigma) = \min\{1, U_R(\mathbf{Y}; \mathbf{y}, \mu, \sigma)\}, \tag{9}$$

where $U_R(\mathbf{Y}; \mathbf{y}, \mu, \sigma)$ is as defined in (8). Clearly, the distribution of U^* is independent of (μ, σ) and $U^*(\mathbf{y}; \mathbf{y}, \mu, \sigma) = R(\tau; \mu, \sigma)$. The exact cdf of U^* is derived in the following subsection.

3.1 Distribution of the generalized pivot

To find the cdf of U^*, we consider two different cases if $\tau \leq \hat{\mu}_0$ or $\tau > \hat{\mu}_0$. Notice that in the case of $\tau \leq \hat{\mu}_0$, the U^* defined in (9) is a mixed random variable with probability function

$$f_{U^*}(x) = \begin{cases} f_{U_R}(x), & 0 < x < 1, \\ \pi, & x = 1, \end{cases}$$

such that

$$\pi = \mathrm{P}(U_R \geq 1) = \mathrm{P}\left(Z_1 + \frac{\tau - \hat{\mu}_0}{\hat{\sigma}_0} Z_2 \leq 0\right), \tag{10}$$

where Z_1 and Z_2 are as defined in (4). So, by some algebraic calculations, we get $\pi = 1 - \left(1 - \frac{n(\tau-\hat{\mu}_0)}{m\hat{\sigma}_0}\right)^{1-m}$. Therefore, the cdf of U^* is given by

$$F_{U^*}(x) = \begin{cases} 0, & x \leq 0, \\ x^n \left(1 - \frac{n(\tau-\hat{\mu}_0)}{m\hat{\sigma}_0}\right)^{1-m} \leq 1 - \pi, & 0 < x < 1, \\ 1, & x \geq 1. \end{cases} \tag{11}$$

It is obvious that in the case of $\tau > \hat{\mu}_0$, the U^* defined in (9) is a continuous random variable; that is, $U^* = U_R$. Hence, by performing some algebraic calculations, the cdf of U^* in this case is given by

$$F_{U^*}(x) = \begin{cases} 0, & x \leq 0, \\ \phi(x; \tau, \hat{\mu}_0, \hat{\sigma}_0), & 0 < x < 1, \\ 1, & x \geq 1, \end{cases} \tag{12}$$

such that

$$\begin{aligned} \phi(x; \tau, \hat{\mu}_0, \hat{\sigma}_0) &= \frac{\Gamma\left(m-1, -\frac{m\hat{\sigma}_0 \log x}{\tau - \hat{\mu}_0}\right)}{\Gamma(m-1)} \\ &+ \frac{x^n}{\Gamma(m-1)} \psi\left(-\frac{m\hat{\sigma}_0 \log x}{\tau - \hat{\mu}_0}, \left(1 - \frac{n}{m}\frac{\tau - \hat{\mu}_0}{\hat{\sigma}_0}\right), m-1\right), \end{aligned}$$

where $\Gamma(\alpha)$ stands for the complete gamma function, $\Gamma(\alpha, t)$ represents the incomplete gamma function (*i.e.*, $\Gamma(\alpha, t) = \int_t^\infty e^{-y} y^{\alpha-1} dy$) and

$$\psi(t, \beta, \alpha) = \begin{cases} \frac{t^\alpha}{\alpha}, & if\ \beta = 0,\ t > 0,\ \alpha > 0, \\ \frac{\Gamma(\alpha) - \Gamma(\alpha, \beta t)}{\beta^\alpha}, & if\ \beta \neq 0,\ t > 0,\ \alpha > 0. \end{cases}$$

3.2 Interval estimation for survival function

As previously mentioned, the percentiles of U^* construct the GCIs for the survival function at τ. Using (6), for given α, a $100(1-\alpha)\%$ lower GCL for $R(\tau; \mu, \sigma)$ is defined by $\inf\{x : F_{U^*}(x) \geq \alpha\}$. To derive the exact lower GCLs for $R(\tau; \mu, \sigma)$, we consider two cases whether $\tau \leq \hat{\mu}_0$ or $\tau > \hat{\mu}_0$.

Case I) Suppose that $\tau \leq \hat{\mu}_0$, then using (11), the α-quantile of U^* is defined by $F_{U^*}^{-1}(\alpha)$ if $\alpha < 1 - \pi$ and coincides with 1 otherwise. Hence, a $100(1-\alpha)\%$ lower GCL for $R(\tau; \mu, \sigma)$ is given by

$$R_1(\tau; \alpha) = \min\left\{1, \alpha^{1/n}\left(1 - \frac{n(\tau - \hat{\mu}_0)}{m\hat{\sigma}_0}\right)^{\frac{m-1}{n}}\right\}. \tag{13}$$

Case II) Now, suppose that $\tau > \widehat{\mu}_0$, then a $100(1-\alpha)\%$ lower GCL for $R(\tau; \mu, \sigma)$ is given by

$$R_2(\tau; \alpha) = F_{U^*}^{-1}(\alpha), \tag{14}$$

where $F_{U^*}(x)$ is as defined in (12). The lower GCLs for $R(\tau; \mu, \sigma)$ can be obtained by FindRoot command in Mathematica, using (14).

To illustrate the performance of the proposed procedure in this paper, we simulate the values of 95% lower GCLs for survival function at τ for $n = 10$, $m = 5$ and some selected choices of the progressive censoring schemes $\mathbf{R} = (R_1, \ldots, R_m)$. Furthermore, the values of μ and σ have been chosen to be $\mu = 0.5, 1$ and $\sigma = 0.4, 1.1$. The results are presented in Table 1. Similar results are tabulated in Table 2 for the associated coverage probabilities. The lower GCLs and the coverage probabilities are obtained using 5000 times simulations.

Table 1. Values of 95% lower GCLs for $R(\tau; \mu, \sigma)$ for $n = 10$ and $m = 5$.

		$\mu = 0.5$				$\mu = 1$		
		τ				τ		
σ	**R**	1	1.5	2	5	1.5	2	5
0.4	(1,1,1,1,1)	0.1017	0.0181	0.0042	0.0000	0.1030	0.0186	0.0000
	(0,0,0,0,5)	0.1015	0.0183	0.0044	0.0000	0.1030	0.0184	0.0000
	(0,5,0,0,0)	0.1050	0.0194	0.0047	0.0000	0.1028	0.0187	0.0000
	(5,0,0,0,0)	0.1053	0.0193	0.0047	0.0000	0.1031	0.0188	0.0000
	(0,0,3,2,0)	0.1037	0.0191	0.0046	0.0000	0.1041	0.0191	0.0000
1.1	(1,1,1,1,1)	0.3940	0.1760	0.0859	0.0031	0.3992	0.1797	0.0052
	(0,0,0,0,5)	0.3920	0.1747	0.0851	0.0031	0.3986	0.1790	0.0052
	(0,5,0,0,0)	0.3978	0.1789	0.0874	0.0032	0.3954	0.1777	0.0050
	(5,0,0,0,0)	0.3940	0.1752	0.0855	0.0031	0.3935	0.1753	0.0048
	(0,0,3,2,0)	0.3972	0.1779	0.0867	0.0031	0.3985	0.1783	0.0050

Table 2. Coverage probabilities of the lower GCLs for $R(\tau; \mu, \sigma)$ for $n = 10$ and $m = 5$.

		$\mu = 0.5$				$\mu = 1$		
		τ				τ		
σ	**R**	1	1.5	2	5	1.5	2	5
0.4	(1,1,1,1,1)	95.30	95.48	95.44	95.48	95.04	94.90	94.90
	(0,0,0,0,5)	95.00	95.06	95.06	94.90	94.90	95.12	95.04
	(0,5,0,0,0)	94.54	94.50	94.58	94.62	95.22	95.36	95.18
	(5,0,0,0,0)	94.72	94.58	94.70	94.76	95.36	95.28	95.30
	(0,0,3,2,0)	94.56	94.72	94.82	94.78	95.00	95.02	95.12
1.1	(1,1,1,1,1)	95.30	95.39	95.06	94.80	94.96	95.02	94.84
	(0,0,0,0,5)	95.62	95.54	95.22	94.96	95.16	94.72	94.54
	(0,5,0,0,0)	94.58	94.82	94.78	94.88	95.22	95.32	95.48
	(5,0,0,0,0)	95.16	95.20	94.98	95.04	95.08	95.16	95.38
	(0,0,3,2,0)	94.98	94.98	95.00	95.10	94.64	94.60	95.12

From Table 2, it is observed that the coverage probabilities of the GCLs are satisfactory.

4 Concluding remarks

The interval estimation of the survival function of the two-parameter exponential distribution on the basis of the progressively Type-II censored samples was studied in this paper. Toward this end, we obtained the GCI on the basis of a generalized pivotal quantity for the survival function. One may also derive a GCI which the associated expected width is minimum. The interval (L, U) is called a $100(1-\alpha)\%$ CI with the shortest expected width for unknown parameter θ on the basis of the pivotal quantity Q, if $F_Q(U) - F_Q(L) = 1 - \alpha$ and $\mathrm{E}(U - L)$ is minimum. Since the probability function of U^* defined in (9), for the case of $\tau \leq \hat{\mu}_0$, is increasing, the $100(1-\alpha)\%$ GCI with the shortest expected width for $R(\tau; \mu, \sigma)$ is $[R_1(\tau; \alpha), 1]$, where, the $R_1(\tau; \alpha)$ is as defined in (13). Now, suppose that $\tau > \widehat{\mu}_0$, then using (12), it can be shown that $f_{U^*}(0) = f_{U^*}(1) = 0$ and that the probability density function of U^* defined in (9) has a unique mode in $(0, 1)$. Therefore, by some algebraic calculations it is deduced that the interval $(\zeta_\alpha, \xi_\alpha)$ is a $100(1-\alpha)\%$ GCI with the shortest expected width for $R(\tau; \mu, \sigma)$ on the basis of the generalized pivot U^*, if

$$f_{U^*}(\zeta_\alpha) = f_{U^*}(\xi_\alpha) \quad \text{and} \quad F_{U^*}(\xi_\alpha) - F_{U^*}(\zeta_\alpha) = 1 - \alpha.$$

The exact values of ζ_α and ξ_α can be easily obtained using the FindRoot command in Mathematica.

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Impact of measured confounders in estimating the causal effect of an exposure: a Monte Carlo study

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Abstract

Controlling measured confounders to estimate the average causal effect of an exposure in observational study is challenging subject. Inverse Probability Treatment Weight (IPTW) estimator method control the impact of these covariates by allocating weight to each subject. In this work, we use Monte Carlo simulation to produce different kind of weights and determine which of the impressive covariates has more effect on the causal path way from exposure to outcome.

Keywords and Phrases: Balanced weights, IPTW, Monte Carlo, Unstablized weights
AMS Subject Classification 2000: primary 62N01.

1 Introduction

Randomized allocation in randomized controlled trials remarks this study as the gold standard method for estimating treatment effect. Hence there are some instances which this study is impractical or unethical, the interest of using observational study increased gradually during past decades. Although we could not guarantee the consistency of causal estimate in this study because of no control in assignment treatment to subjects which result in systematically different between treated and untreated subjects[1] and presence of measured confounding [2], but so called IPTW estimation method can reduce the impact of confounder's by assigning weights to each subject. These weights could be stablized, unstablized or balanced. In this papaer we consider different covariates which affect on exposure and outcome. Then by Monte Carlo study we obtain stablized and balanced weights and determine how the impact of covariates under different models can be controlled.

2 IPTW Estimator

For controlling the effect of measured confounders, IPTW estimator create a Pseudo population by assinging wieght to each subject. By each weight we have in fact copies from each subject in puasode population. For instance if the weight of i^{th} subject be 2 then we have 2 copies from this subject in pseudo population. Pseudo population has 2 characters[3]:

1) Treatment or exposure in pseudo population is not influence by measured confounders.

2) $P[Y_{i1} = 1]$ and $P[Y_{i0} = 1]$ are the same in basic and pseudo populations.

Y_{i1} and Y_{i0} are called potential outcomes. Suppose X in a binary exposure (X_i=0: unexposed, X_i=1: exposed) then Y_{ix} represent the potential outcome which a subject would have had if possibly contrary to fact, the treatment X were set to x.[4]

For IPTW estimator the weights can be obtained as follow:

$$W_i = 1/P[X_i|U_i]$$

where X is an exposure and U is measured confounder factors.

These probabilities are unknown and should be estimated. For instance, if the exposure be binary we can postulate a logistic regression as the form:

$$Logit(X_i|U_i) = \alpha_0 + \alpha_1 U_i$$

where $logit(z) = \frac{exp(z)}{1+exp(z)}$

These weights are called unstablized. We can use stablized weights instead to obtain more consistent estimate of the causal effect. Stablized weights are difined as[3]:

$$SW_i = P[X_i]/P[X_i|U_i]$$

When confounders and exposure be independent then $SW_i = 1$ so each subject'weight is the weight in basic population, that is why we have more consistent estimator under stablized weights.

For calculating balanced weights, fisrt we regress the exposure on measured confounders to obtain fitted value of exposure then we obtain balanced weights as follow:

$$BW_i = 1/(X_i X_i^* + (1 - X_i)(1 - X_i^*))$$

where X_i^* is the fitted value of X_i.

3 Monte Carlo Simulation

3.1 definitions

Seven variables considered which varying in their association with the exposure and the outcome. X_2,X_4,X_5,X_6 have been generated independently from normal distribution with mean 0 and standard deviation 1. X_1, X_3 and X_7 have been generated from bernoulli distribution with parameter 0.5 . X_1, X_2, X_5 and X_6 are associated with both exposure and outcome. X_4 is associated with outcome only and X_3 is associated with exposure only. X_7 is associated with neither exposure nor outcome. Based on epidemiologist standpoint[1], just 4 of these variables are true confounders that is they affect on both exposure and outcome. Other variables are baseline covariates because they don't have effect on exposure and outcome simultaneously, in addition they are measured before starting the study.

we used logit model to generate exposure as follow:

$$logit(p_{i,w}) = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_4 x_5 + \beta_5 x_6$$

$W_i : Ber(1, p_{i,w})$

The outcome is generated from normal distribution with $\mu = \alpha_0 + \alpha_1 x_1 + \beta_2 x_2 + \alpha_3 x_4 + \alpha_4 x_5 + \alpha_5 x_6$ and $\sigma = 1$.

By Monte Carlo simulation we determine the ability of different case of IPTW estimators to balance seven covariates between subjects. We considered 5 models by different choice of variables entering the model. In first model, we just entered true confounders to obtain stablized and balance wieght. In second model, we entered X_3 which has correlation with exposure. In third model, we use X_4 that has effect on outcome. Fourth model has X_7 that does not have effect on exposure and outcome. In the last model we omitted one of the true confounders. we wish to determine which of the covariates has more effect on the causal effect of an exposure.

IPTW1: (X_1, X_2, X_5, X_6)
IPTW2: $(X_1, X_2, X_5, X_6, X_3)$
IPTW3: $(X_1, X_2, X_5, X_6, X_4)$
IPTW4: $(X_1, X_2, X_5, X_6, X_7)$
IPTW5: (X_1, X_2, X_5)

3.2 Monte Carlo simulation

We randomly generated 500 data sets as described in section 3.1 which each data set has 500 subjects. Using each of the 500 data sets, we obtain the bias and mean square error (MSE) of the estimated causal effect for stablized and balanced weights under Monte Carlo and general Simulation.

3.3 Result

We summarize our findings by considering table 1 and table 2:

First, stablized weights result in better estimate of causal effect than balanced weight in Monte Carlo simulation whereas it is vice versa in general simulation.

Second, entering X_4 to the model which has effect on outcome only reduced bias and MSE in both study it means by adding X_4, we could estimate the causal effect more consistently in comparison with ignoring this variable. It seeme that entering X_3 would not have effect on the causal path way from exposure to an ouctome.

Third, entering X_7 which does not have effect on both exposure and outcome decrease the bias in Monte Calo study but these factors doesnt change in general simulation. This result is more logical than Monte Carlo simulation because each covariate which has effect nor exposure neither outcome could not control the effect of confounders in the causal path way.

Finally, omitting one of true confounders lead to more bias. hence confounders affect on both exposure and outcome, ignoring their effect deviate the causal effect of an exposure in an outcome.

4 Discussion

The objective of the current study was to determine the impact of different covariates on the causal effect of an exposure on an outcome under Monte Carlo simulation and general simulation. By considering table 1 and table 2 we figur out that entering different covariates to the analysis have different consequences. In addition we could not determine which of the weights has more efficieny in IPTW estimator. Depend on which simululation performed we should choose weights. But an obviouse result is ignoring the impact of one of the confounders deviate the causal effect. That is why controlling measured confounders is essential especially in epidemiology. In this study we considered estimation models in which the espocure is affecting linear on outcome.

Table1. IPTW Estimator with Monte Carlo Simulation

Estimator:	Bias		MSE	
weights:	Balanced	Stablized	Balanced	Stablized
IPTW1	0.333	0.108	0.248	0.028
IPTW2	0.302	0.109	0.207	0.028
IPTW3	0.291	0.106	0.182	0.027
IPTW4	0.311	0.108	0.229	0.028
IPTW5	0.341	0.106	0.253	0.027

Table2. IPTW Estimator by General Simulation

Estimator:	Bias		MSE	
weights:	Balanced	Stablized	Balanced	Stablized
IPTW1	0.080	0.108	0.019	0.028
IPTW2	0.080	0.108	0.019	0.028
IPTW3	0.076	0.105	0.018	0.027
IPTW4	0.080	0.108	0.019	0.028
IPTW5	0.098	0.106	0.025	0.027

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Residual Renyi entropy properties of record values

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Abstract

In this paper,we explore properties of the residual Renyi entropy of upper (lower) record variables. The residual Renyi entropy of the *nth* record from a continuous distribution function is represented in terms of the residual Renyi entropy of the *nth* record from exponential distribution and closed form of incomplete gamma distribution. We discuss the monotone behavior of the residual Renyi entropy of record values and provide bounds for residual Renyi entropy of upper (lower) records.

Keywords and Phrases: Upper record values, Lower record values, Residual life time, Renyi entropy.

AMS Subject Classification 2000: primary 94A17; secondary 62G30.

1 Introduction

The notation of entropy is of fundamental importance in different areas such as physics, probability and statistics, communication theory and economics. Shannon entropy plays an important role in the context of information theory. Let X be an absolutely continuous random variable which denotes the lifetime of a device or, a system with probability density function $f(x)$, distribution function F and the survival function $\bar{F} = 1 - F$. Then the average amount of uncertainty associated with the random variable X, as given by the Shannon entropy (1948), is:

$$H(X) = H(F) = -\int_0^{+\infty} f(x) log f(x) dx.$$

$$H_\alpha(X) = -\frac{1}{\alpha - 1} log \int_0^{+\infty} f^\alpha(x) dx, \alpha > 0, \alpha \neq 1.$$

Ebrahimi [11] considered the entropy of the residual lifetime as a dynamic measure of uncertainty. The residual lifetime of the system when it is still operating at time t, is $X_t = X - t | X > t$ which has probability density function

$$f(x;t) = \frac{f(x)}{\bar{F}(t)}, x > t > 0, \bar{F}(t) > 0.$$

This function can also be used to describe the mean residual life time

$$H(X;t) = -\int_t^{+\infty} f(x;t) log f(x;t) dx, t > 0.$$

The residual entropy is time-dependent and measures the uncertainty of the residual lifetime of the system when it is still operating at time t. Several authors have studied properties of $H(X;t)$;see for example, Ebrahimi and Kirmani [12], Asadi and Ebrahimi [5] and Belzunce et al. [9].
The residual Renyi entropy (RRE) is defined similarly:

$$H_\alpha(X;t) = -\frac{1}{\alpha - 1} log \int_t^{+\infty} \frac{f^\alpha(x)}{\bar{F}^\alpha(t)} dx. \tag{1}$$

Let $X_i, i \geq 1$, be a sequence of iid continuous random variables with the cdf $F(x)$ and the pdf $f(x)$. An observation X_j will be called an upper record value if its value is greater than that of all previous observations. Thus X_j is an upper record value if $X_j > X_i$ for all $i < j$. By convention X_1 is the first upper record value. An analogous definition can be given for lower record values. Record values and associated statistics are of great importance in several real-life problems involving weather, economic and sports data. The statistical study of record values started with Chandler [10] and has now spread in different directions. Interested readers may refer to Nagaraja [21] and Arnold et al. [6]. Some work in this regard has been done for Rayleigh and Weibull distribution by Balakrishnan and Chan [7] and Sultan and Balakrishnan [20], Habibi et al. [13] compared Kullback Leibler information of records with the same number of *iid* observations. Abbasnejad and Arghami [2] did the same tinging terms of Renyi information. Baratpur et.al. [8] studied some information properties of records based on Shannon entropy and mutual information.

Several ordering and reliability properties of record values have been studied by Kamps [14], [15], Ahmadi and Arghami [3], Ahmadi and Balakrishnan [4], Zarezadeh and Asadi [23]. Navarro and Agulia and Asadi [22]studied some new result on cumulative residual entropy. Kumar and Taneja [17] worked some characterization result on generalized cumulative result entropy measure. We now provide some results on the residual Renyi entropy of records. In section 2, first we express the marginal density and survival function, then present lemma and theorems in order to drive closed forms for RRE of nth record values. We also show that,RRE of upper (lower) records is non-increasing function of number of record values in the sequence. In section 3, we obtain bounds for residual Renyi entropy of records, and we also give figures to show that bounds of RRE of nth record value based on mode of Gamma distribution are closer to real value RRE than bound presented by Zarezadeh and Asadi.

We bring the following definition in which X and Y denote random variables with distribution functions F and G, density functions f and g, and survival functions: $\bar{F}(x) = 1 - F(x), \bar{G}(x) = 1 - G(x)$.

definition 1.1

(a) The random variable Y is said to be smaller than X in the usual stochastic order ($Y <_{st} X$) if $\quad \bar{G}(x) < \bar{F}(x) \quad$ for all x.

(b) The random variable Y is said to be smaller than X in likelihood ratio order ($Y <_{lr} X$) if $\frac{f(x)}{g(x)}$ is an increasing function of x.

It can be shown that if $Y <_{lr} X$, then $Y <_{st} X$. See [19]

2 Residual renyi entropy of upper(lower) record values

Let $U_1, U_2, ..., U_n$ be the first n upper record values from a distribution with the cdf $F(x)$ and the pdf $f(x)$. Then the joint pdf of the first n upper record values and the marginal density of U_n (the nth upper record value , $n \geq 1$) is given , respectively, by

$$q(u) = \prod_{i=1}^{n-1} r(u_i) f(u_n), u_1 < ... < u_n$$

$$f_{u_n}(u_n) = \frac{R^{n-1}(u_n)}{\Gamma(n)} f(u_n), \quad \infty < u_n < +\infty. \tag{2}$$

Where $R(t) = -log(1 - F(t))$, and $r(t) = R'(t) = \dfrac{f(t)}{1 - F(t)}$is the hazard function. The survival function of U_n, which is denoted by $\bar{F}_{U_n}$,is given by

$$\bar{F}_{U_n}(x) = \sum_{j=o}^{n-1} \frac{[-log\bar{F}(x)]^j}{j!} \bar{F}(x) = \frac{\Gamma(n; -log\bar{F}(x))}{\Gamma(n)}. \tag{3}$$

Where $\Gamma(a; x)$ is known as the incomplete Gamma function and is defined as

$$\Gamma(a; x) = \int_x^{+\infty} u^{a-1} e^{-u} du, \quad a, x > 0.$$

$$Where \ \ f_t(x) = \frac{1}{\Gamma(a;t)} x^{a-1} e^{-x}, \qquad x > t > 0. \tag{4}$$

Let $L_1, L_2, ..., L_n$ be the first n lower records values from the distribution with the *cdf* $F(x)$ and the pdf $f(x)$.Then the joint pdf of the first n lower record values and the marginal density of L_n (the *nth* lower record value, $n \geq 1$)is given , respectively by:

$$p(I) = \prod_{i=1}^{n-1} \tilde{r}(l_i) f(l_n), \quad l_1 > ... > l_n$$

$$f_{l_n}(l_n) = \frac{\tilde{R}^{n-1}}{(n-1)!} f(l_n), \qquad -\infty < L_n < +\infty, \tag{5}$$

Where $\tilde{R}(t) = -log(F(t)), and \ \tilde{r}(t) = \dfrac{f(t)}{F(t)}$.
Is the reversed hazard function. Then the survival function of L_n, which is denoted by $\bar{F}_{L_n}$,is given by

$$\bar{F}_{L_n}(x) = 1 - \sum_{j=o}^{n-1} \frac{[-logF(x)]^j}{j!} F(x) = \frac{\Gamma^*(n; -logF(x))}{\Gamma^*(n)}. \tag{6}$$

where $\Gamma^*(a;x)$ is known as the incomplete Gamma function and is defined as:

$$\Gamma^*(a;x) = \int_0^x u^{a-1} e^{-u} du, \quad a, x > 0.$$

In this section we obtain some results on the RRE of record values.
First, we explore the following lemma witch obtain RRE of record values from standard Exponential distribution in terms of truncated Gamma distribution. It is easily proved from definition of *RRE*.
Lemma 2.1. Let U_n^*denote the *nth* upper record value from a sequence of observations from standard Exponential distribution. Then

$$H_\alpha(U_n^*; t) = -\frac{1}{\alpha-1} log \frac{\Gamma(\alpha(n-1)+1, t)}{\Gamma^\alpha(n,t)} - \frac{1}{\alpha-1} log E(e^{-(\alpha-1)U_t}) \ \ . \tag{7}$$

where $U_t \sim \Gamma(\alpha(n-1)+1, t)$.
Theorem 2.1. Let$X_n, n > 1$ be a sequence of *iid* continuous random variables from the distribution $F(x)$ with density function $f(x)$ and the quantile function $F^{-1}(.)$. Let U_n denote the nth upper record. Then the Renyi entropy of U_n can be expressed as

$$H_\alpha(U_n; t) = -\frac{1}{\alpha-1} log \frac{\Gamma(\alpha(n-1)+1, -log\bar{F}(t)))}{\Gamma^\alpha(n, -log\bar{F}(t))} - \frac{1}{\alpha-1} log E(f^{\alpha-1}(F^{-1}(1-e^{-V_z})). \tag{8}$$

Also we can write:

$$H_\alpha(U_n; t) = H_\alpha(U_n^*, -log\bar{F}(t)) + \frac{1}{\alpha-1} log E(e^{-(\alpha-1)V_z}) - \frac{1}{\alpha-1} log E(f^{\alpha-1}F^{-1}(1-e^{-V_z}). \tag{9}$$

Where $z = -log\bar{F}(t)$ and $V_z \sim \Gamma(\alpha(n-1)+1, -log\bar{F}(t))$.

Proof. By formula(1), (2), (3), and applying transformation $v = R(x)$ we have

$$\begin{aligned}
H_\alpha(U_n; t) &= -\frac{1}{\alpha-1} log \int_{-log\bar{F}(t)}^{+\infty} \frac{v^{\alpha(n-1)} e^{-v}}{\Gamma(n)^\alpha} f^{\alpha-1}(F^{-1}(1-e^{-v}))dv + \frac{\alpha}{\alpha-1} log\bar{F}_{U_n}(t) \\
&= \frac{\alpha}{\alpha-1} log\Gamma(n) - \frac{1}{\alpha-1} log\Gamma(\alpha(n-1)+1, -log\bar{F}(t)) \\
&- \frac{1}{\alpha-1} log \int_{-log\bar{F}(t)}^{+\infty} \frac{v^{\alpha(n-1)} e^{-v}}{\Gamma(\alpha(n-1)+1, -log\bar{F}(t))} f^{\alpha-1}(F^{-1}(1-e^{-v}))dv \\
&+ \frac{\alpha}{\alpha-1} log \frac{\Gamma(n, -log\bar{F}(t))}{\Gamma(n)} \\
&= -\frac{1}{\alpha-1} log \frac{\Gamma(\alpha(n-1)+1, -log\bar{F}(t))}{\Gamma(n, -log\bar{F}(t))^\alpha} - \frac{1}{\alpha-1} log E(f^{\alpha-1}(F^{-1}(1-e^{-V_z})).\square
\end{aligned}$$

Theorem 2.2. Under the assumptions of theorem 2.1, if L_n denotes the nth lower record, then the Renyi entropy of L_n can be expressed as

$$H_\alpha(L_n;t) = -\frac{1}{\alpha-1} log \frac{\Gamma^*(\alpha(n-1)+1; -logF(t))}{\Gamma^{*\alpha}(n, -logF(t))} - \frac{1}{\alpha-1} logE(f^{\alpha-1}(F^{-1}(e^{-V_z})). \tag{10}$$

Where $V_z \sim \Gamma^*(\alpha(n-1)+1, -logF(t))$.
Proof. By using (1),(3),(5), Where $v = \tilde{R}(t) = -log(F(t))$, it can be proved in a similar manner of theorem 2.1. □

Lemma 2.2.(Shaked, M., 2007) The expectations $X \leqslant_{st} Y$ exist if and only if $E[\varphi(X)] < E[\varphi(Y)]$ holds for all increasing functions φ for which the expectations exist.
Theorem 2.3. Let $X_i, i > 1$ be a sequence of i.i.d random variables from distribution function F having an increasing density function$f$. If $U_n, n > 1$ represents the sequence of upper record values corresponding to F, then $H_\alpha(U_n;t)$ is decreasing in n.
Proof. Upon using closed form of $H_\alpha(U_n;t)$ we can write:
$H_\alpha(U_n;t) = \delta(n,t) - \frac{1}{\alpha-1} logE(f^{\alpha-1}(F^{-1}(1-e^{-V_n})),$
Where $\delta(n,t) = -\frac{1}{\alpha-1} log(\Gamma(\alpha(n-1)+1, -log\bar{F}(t))) - \frac{\alpha}{\alpha-1}\Gamma(n, -log\bar{F}(t))$.
$H_(\alpha)(U_{n+1};t) - H_\alpha(U_n;t) = \delta(n+1,t) - \delta(n,t) - \frac{1}{\alpha-1} log(\frac{E(f^{\alpha-1}(F^{-1}(1-e^{-V_{n+1}})))}{E(f^{\alpha-1}(F^{-1}(1-e^{-V_n})))})$.
Where $V_n \sim \Gamma^*(\alpha(n-1)+1, -logF(t))$.
For $\alpha > 1$ it is enough to show that:
$\frac{d\delta(n,t)}{dn} = \frac{\alpha}{\alpha-1}(\psi(n, -log\bar{F}(t)) - \psi(\alpha(n-1)+1, -log\bar{F}(t)))$
Where $\psi(x) = \frac{dlog\Gamma(x)}{dx}$ is an increasing digamma function, for all $x, \delta(n,t)$ is a decreasing function of n.
And we obtain $V_n \leqslant_{lr} V_{n+1}$ therefore $V_n \leqslant_{st} V_{n+1}$. By lemma 2.2 :
$E[f^{\alpha-1}(F^{-1}(1-e^{-v_n}))] > E[f^{\alpha-1}(F^{-1}(1-e^{-v_{n+1}}))]$.
Thus $H_\alpha(U_{n+1};t) - H_\alpha(U_n;t) < 0$, and result for $0 < \alpha < 1$ readily follow.□

Theorem 2.4. Let $X_i, i > 1$ be a sequence of i.i.d random variables from distribution function F having an decreasing density function f. If $L_n, n > 1$ represents the sequence of lower record values corresponding to F, then $H_\alpha(L_n;t)$ is decreasing in n.
Proof. Using argument similar to those in the proof of theorem 2.3 we can prove it. □

3 Bounds for Residual Renyi entropy of upper(lower)record values

Zarezadeh and Asadi (2010) obtained bound for residual Renyi entropy of upper record values. In this section, we introduce bounds for *RRE* of *nth* record a s different distribution in terms of mode of Gamma distribution and mode of the parent distribution.

Theorem 3.1. For any random variable X with Renyi entropy $H_\alpha(X;t) < \infty$ the Renyi entropy of upper record $U_n, n = 1, 2, ...,$ is bounded as follows:
for $\alpha > 1 (0 < \alpha < 1)$;

$$H_\alpha(U_n;t) > (<) + \frac{\alpha}{\alpha-1} log\Gamma(n, -log\bar{F}(t)) - \frac{1}{\alpha-1}(\alpha(n-1)log(m_n) - m_n) + S(t) \tag{11}$$

where $\quad S(t) = -\frac{1}{\alpha-1} log \int_t^\infty r(y) f^{\alpha-1}(y)dy, \quad r(y) = \frac{f(y)}{\bar{F}(y)}$

Proof. Let $H_\alpha(U_n;t) < \infty$, if $m_n = max\{\alpha(n-1), -log\bar{F}(t)\}$
where m_n is the mode of Gamma distribution with density:

$$M_n = f(m_n) = \frac{m_n^{\alpha(n-1)} e^{-m_n}}{\Gamma(\alpha(n-1)+1, -log\bar{F}(t))}.$$

Now, we write ,for $\alpha > 1 (0 < \alpha < 1)$

$$\begin{aligned}
& - \frac{1}{\alpha-1} logE(f^{-1}(F^{-1}(1-e^{-V})) \\
&= -\frac{1}{\alpha-1} log \int_{-log\bar{F}(t)}^{\infty} \frac{v^{\alpha(n-1)}e^{-v}}{\Gamma(\alpha(n-1)+1, -log\bar{F}(t)} (f^{\alpha-1}(F^{-1}(1-e^{-v}))dv, \\
&> -\frac{1}{\alpha-1} logf(m_n) - \frac{1}{\alpha-1} log \int_{-log\bar{F}(t)}^{+\infty} (f^{\alpha-1}(F^{-1}(1-e^{-v}))dv, \\
&= -\frac{1}{\alpha-1} logM_n + \frac{-1}{\alpha-1} log \int_t^{+\infty} r(y) f^{\alpha-1}(y)dy. \square
\end{aligned}$$

By transformation $y = F^{-1}(1-e^{-v})$ and $r(y) = \frac{f(y)}{\bar{F}(y)}$ above result obtain. Therefore

$$H_\alpha(U_n;t) < + \frac{\alpha}{\alpha-1} log\Gamma(n, -log\bar{F}(t)) - \frac{1}{\alpha-1}(\alpha(n-1)logm_n - m_n) + S(t).$$

Thus the proof is now complete. $\square$

Theorem 3.2. For any random variable$X$ with Renyi entropy $H_\alpha(X;t) < \infty$ the Renyi entropy of lower record $L_n, n = 1, 2, ...,$ is bounded as follows:
For $\alpha > 1 (0 < \alpha < 1)$;

$$H_\alpha(L_n;t) > (<) + \frac{\alpha}{\alpha-1} log\Gamma(n, -logF(t)) - \frac{1}{\alpha-1}(\alpha(n-1)log(m_n) - m_n) + S^*(t). \tag{12}$$

where $S^*(t) = -\frac{1}{\alpha-1} log \int_t^{\infty} r(y) f^{\alpha-1}(y)dy, \quad r(y) = \frac{f(y)}{F(y)}$
Proof. The similar argument proof of theorem 3.1 using for prove it.$\square$

Zarezadeh and Asadi (2010) proposed using of the following bound for RRE of upper record values.

$$H_\alpha(U_n;t) > -\frac{1}{\alpha-1} log \frac{\Gamma(\alpha(n-1)+1, -log\bar{F}(t))}{\Gamma^\alpha(n, -log\bar{F}(t))} - logM \tag{13}$$

We obtain bound for RRE of lower record values the based on mode of the parent distribution.

Theorem 3.3. Under the assumptions of Theorem 3.2, for $\alpha > 0$;

$$H_\alpha(L_n;t) > -\frac{1}{\alpha-1} log \frac{\Gamma^*(\alpha(n-1)+1, -logF(t))}{\Gamma^{*\alpha}(n, -logF(t))} - logM, \tag{14}$$

where $M = f_X(m) < \infty$, and $m = sup\{X : f(x) < M\}$ is the mode of the distribution.
Proof. The proof is easy. Since $f_X(x) < f_X(m)$, it is enough to show for $\alpha > 1 (0 < \alpha < 1)$:
$f^{\alpha-1}F^{-1}(1-e^{-v}) < (>) M^{\alpha-1}$,
We have: $\frac{-1}{\alpha-1} logE[f^{\alpha-1}F^{-1}(1-e^{-v})] > -logM.$
Thus the result follows.$\square$

Example 3.2. Let X have Generalized exponential distribution with density
$f(x) = \frac{\beta}{\theta} exp(\frac{\mu-x}{\theta})(1-exp(\frac{\mu-x}{\theta}))^{\beta-1}, \qquad x < \mu < 0 \quad \theta, \beta > 0,$

since the mode of distribution is $m = \mu + \theta ln\beta$ we have :
$M = f(m) = \frac{1}{\theta}(\frac{\beta-1}{\beta})^{\beta-1}$,
therefore,
$$H_\alpha(L_n; t) > -\frac{1}{\alpha-1} log \frac{\Gamma^*(\alpha(n-1)+1, -logF(t))}{\Gamma^{*\alpha}(n, -logF(t))} + log\theta - (\beta-1)log(1-\frac{1}{\beta}).$$

Remark 3.1. The following figures show that the bound of RRE of nth upper record value based on mode of Gamma distribution in (11), is more exact than the bound of presented by Zarezadeh and Asadi in 15. Figures display plot of the RRE of U_n (black line) and bounds in 11 (violet line), in 15 (blue line) for Gamma distribution and Weibull distribution.

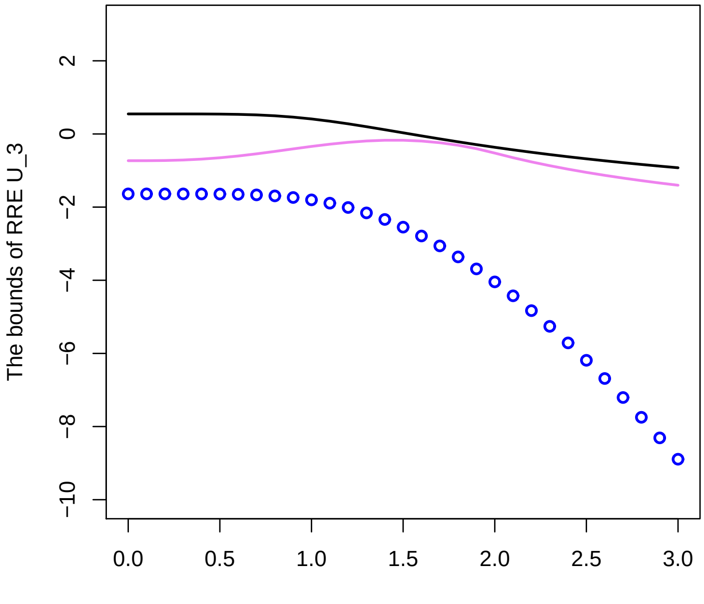

Figure 1: The plot of the bounds of RRE U_3 for Weibull distribution based on$\alpha = 2, \beta = 2, \lambda = 1$

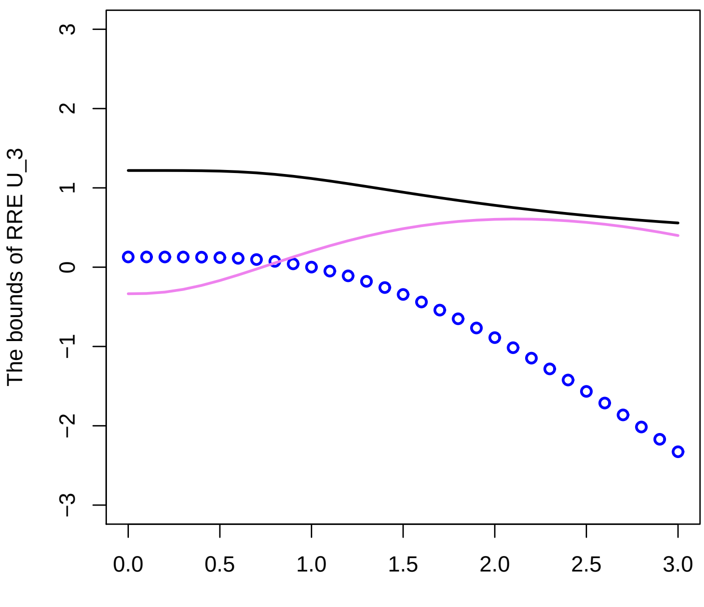

Figure 2: The plot of the bounds of RRE U_3 for Gamma distribution based on$\alpha = 2, \beta = 2, \theta = 2$

4 conclusion

In this paper, we explored properties of the residual Renyi entropy (RRE) of upper (lower) record values. The RRE of the nth record from a continuous distribution function is represented in terms of the residual Renyi entropy of the nth record from exponential distribution and closed form of truncated Gamma distribution. We introduced bounds for RRE of nth record as different distribution in terms of mode of the Gamma distribution and mode of the parent distribution. We showed that bounds of RRE of nth record value based on mode of Gamma distribution are closer to real value RRE than bounds presented on mode of parent distribution.

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